Topology of Synthetic Schemes

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Abstract

The following is a collection of results on topological properties of synthetic schemes. Authors so far: Ingo Blechschmidt, Felix Cherubini, Hugo Moeneclaey, Matthias Hutzler, Marc Nieper-Wißkirchen, David Wärn.

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Introduction

1 Topology of schemes

This collection of topological properties is not meant to be complete. Reduced schemes are defined in the work in progress draft [Che+23a].

1.1 Infinitesimal Neighborhoods

There is an operation, which extends a closed subtype to some infinitesimal extend.

Definition 1.1.1 Let $X = \operatorname{Spec} A$ be affine and $C = \operatorname{Spec} A/I$ a closed subtype given by an finitely generated ideal $I \subseteq A$. Then the *n-th infinitesimal neighborhood* of C in X is the closed subtype

$$C^{n+1} :\equiv \operatorname{Spec} A/(I^{n+1}) \subseteq \operatorname{Spec} A.$$

(TODO: * emphasize pointwise perspective more (operation on closed propositions))

Explicitly, I^k, with $I = (f_1, \dots, f_m)$, is the ideal

$$I^k :\equiv \left\{ \sum_{i=1}^{m} \alpha_i x_{i,1} \cdot \dots \cdot x_{i,k} \mid x_{i,j} : I, \alpha_i : A \right\}.$$

So, by calculation, $(f_1, \dots, f_m)^k$ is generated by all k-fold products of the generators of the f_i.

There is an easy way describing the union of all the k-th neighborhoods of a closed subtype, using double negation:

Lemma 1.1.2 (using ??, ??, ??) Let $X = \operatorname{Spec} A$ be affine and $C \subseteq \operatorname{Spec} A$ a closed subtype. Then we have

$$\bigcup_{n:\mathbb{N}} C^n = \neg\neg C.$$

The subtype $\neg\neg C$ is also called the *formal neighborhood* of C.

Proof Let C be given as $\operatorname{Spec} A/(f_1, \dots, f_l)$. Then, for any $x : X$

$$\begin{aligned} \neg\neg C(x) &= \neg\neg(x \in C) \\ &= \neg\neg(\forall i. f_i(x) = 0) \\ &= \forall_i \quad \neg\neg f_i(x) = 0 \\ &= \forall_i \quad f_i(x) \text{ is nilpotent} \end{aligned}$$

– and $C^n(x)$ implies $\forall_i \quad f_i^n(x) = 0$. And if $\forall_i \quad f_i^n(x) = 0$, then $C^{nl}(x)$. □

1.2 Connectedness

The following is in conflict with the usual use of the word "connected" in homotopy type theory.

Definition 1.2.1 A pointed type X is called *connected*, if the following equivalent statements hold:

(i) Any function $X \to \text{Bool}$ is constant.

(ii) Any detachable subset is X or $\emptyset$.

Proposition 1.2.2 (using ??, ??) The set $\mathbb{A}^1$ is connected, that is, every function $f : \mathbb{A}^1 \to \text{Bool}$ is constant.

Proof We embed Bool into R as the subset $\{0, 1\} \subseteq R$. (We have $0 \neq 1$ in R by (**??**).) Then we have a function $\widetilde{f} : \mathbb{A}^1 \to R$ and we can assume $\widetilde{f}(0) = 0$. Note that $\widetilde{f}$ is an idempotent element of the algebra $R^{\mathbb{A}^1}$, since all its values are idempotent elements of R. By (**??**), $\widetilde{f}$ is given by an idempotent polynomial $p \in R[X]$ with $p(0) = 0$. But from this follows $p = 0$: we can factorize $p = Xq$ and then calculate $p = p^n = X^n q^n$ to see that all coefficients of p are zero. □

A connected scheme, that is covered by its point and everything except the point, is already trivial.

Corollary 1.2.3 Let X be a connected scheme and

$$\prod_{x:X} x = * \vee x \neq *.$$

Then X is contractible.

Proof Assume $\prod_{x:X} x = * \vee x \neq *$. Since for any proposition P, $P + \neg P$ is a proposition, we have $\prod_{x:X} x = * + x \neq *$ and there is a map to Bool from any binary copdoduct. So we have a map $X \to \text{Bool}$ which decides if a general $x : X$ is the point $*$ or not. By connectedness of X, this map is constant, but we know $* = *$, so $x = *$ for all x. □

Corollary 1.2.4 (using ??, ??) $\neg(\prod_{x:\mathbb{A}^1} x = 0 \vee x \neq 0)$

Proof By corollary 1.2.3 and by the connectedness of $\mathbb{A}^1$ (proposition 1.2.2), we can show from $\prod_{x:\mathbb{A}^1} x = 0 \vee x \neq 0$ that $\mathbb{A}^1$ is contractible. This contradicts $1 \neq 0$. □

Example 1.2.5 The ring R is a local ring, so we have $\Pi_{x \in R} \|\text{inv}(x) \vee \text{inv}(1 - x)\|$, but we can prove that the statement without the propositional truncation is false:

$$\neg \Pi_{x \in R}(\text{inv}(x) \amalg \text{inv}(1 - x)).$$

Namely, a witness of $\Pi_{x \in R}(\text{inv}(x) \amalg \text{inv}(1 - x))$ is equivalently a function $f : R \to \text{Bool}$ with the property that

$$\text{if } f(x) \text{ then } \text{inv}(x) \text{ else } \text{inv}(1 - x).$$

But by *proposition* 1.2.2, the function f must be constant, contradicting the fact that $\neg\text{inv}(x)$ for $x = 0$ and $\neg\text{inv}(1 - x)$ for $x = 1$.

In particular, not every type family $B : \mathbb{A}^1 \to \mathcal{U}$ with $\Pi_{x:\mathbb{A}^1} \|B(x)\|$ merely admits a choice function $\Pi_{x:\mathbb{A}^1} B(x)$.

1.3 Compactness properties

?? can be read as a compactness property for countable disjoint open coverings of affine schemes, since functions $f : \mathrm{Spec}\, A \to \mathbb{N}$ correspond to decompositions $\mathrm{Spec}\, A = \sum_{n:\mathbb{N}} U_n$, and the subsets $U_n \subseteq X$ are automatically open because they are detachable.

The following example shows that we can not expect all affine schemes to be compact with respect to arbitrary set-indexed open coverings.

Example 1.3.1 For A a finitely presented R-algebra, consider the open cover $(U_i)_{i\in I}$, where the index set is $I = \mathrm{Spec}\, A$ and for each i we set $U_i = \mathrm{Spec}\, A = D(1)$. This indeed covers all points of $\mathrm{Spec}\, A$, since for every $x \in \mathrm{Spec}\, A$ we clearly have $x \in U_x$. To give a finite subcover of this cover, however, means to give a natural number n and a function $\mathrm{Fin}\, n \to \mathrm{Spec}\, A$ with the property that $\mathrm{Spec}\, A$ is empty if $n = 0$. In essence, it means to decide whether $\mathrm{Spec}\, A$ is inhabited or not. We claim that this is not possible for all finitely presented R-algebras:

$$\neg(\Pi_{A:f.p.R\text{-Alg}} \| \mathrm{Spec}\, A \amalg \neg\, \mathrm{Spec}\, A \|).$$

Indeed, for $A = R/(x)$, the proposition $\| \mathrm{Spec}\, A \amalg \neg\, \mathrm{Spec}\, A \|$ means $x = 0 \vee x \neq 0$, and we saw in **??** that this is not true for all $x \in R$.

There is, however, a notion of compactness, which seems to correspond to completeness and therefore also leads to a notion of properness, which is treated in [Che+23b].

1.4 Dense subtypes

Algebraic preparation:

Lemma 1.4.1 If $P : R[X]$ and we have $P \neq 0$, or equivalently, that merely some coefficient of P is non-zero, then P is nilregular.

Proof If P is non-zero, its content $c(P)$ is top. For any $Q : R[X]$ with $P \cdot Q$ nilpotent, by [LQ15][Theorem III.2.1] we also have $c(P \cdot Q) = c(P) \wedge c(Q) = c(Q)$ is bottom. So Q is nilpotent. □

In a lattice, an element a can be called dense if $a \wedge b = \bot$ implies $b = \bot$. We apply this definition to the lattice of open subtypes of a type X, but generalize it to allow for non-open dense subtypes too.

Definition 1.4.2 A subtype $A \subseteq X$ is called *dense*, if for all open subtypes $V \subseteq X$ such that $V \cap A = \emptyset$, we have $V = \emptyset$.

Lemma 1.4.3 Let X be a type.

(a) If $D \subseteq X$ is dense, then $X \neq \emptyset$ implies $D \neq \emptyset$.

(b) Let $D \subseteq X$ be dense and open and let $E \subseteq X$ be dense, then $D \cap E$ is dense.

(c) Let $D \subseteq X$ be dense and $E \subseteq X$ any subtype, then $D \cup E$ is dense.

Proof (a) Assume $D \subseteq X$ is dense and empty. Then $X \cap D = \emptyset$ and by denseness of D, the open subtype $X \subseteq X$ is empty, which contradicts $X \neq \emptyset$.

(b) Let $V \subseteq X$ be open with $V \cap D \cap E = \emptyset$. Since $V \cap D$ is open and E is dense, we get $V \cap D = \emptyset$, and by denseness of D, we get $V = \emptyset$.

(c) Let $V \subseteq X$ be open with

$$\emptyset = V \cap (D \cup E) = (V \cap D) \cup (V \cap E).$$

This implies $V \cap D = \emptyset$, so $V = \emptyset$. □

Being dense is double negation stable — which has the practical implication, that we can "open" double-negated statements when showing denseness.

Proposition 1.4.4 Being dense is $\neg\neg$-stable: if a subtype $D \subseteq X$ is not not dense, then it is dense.

Proof We use general facts about modalities. $V = \emptyset$ is a pointwise negated statement and therefore $\neg\neg$-stable. Since the proposition that D is dense is a $\prod$-type with values in $V = \emptyset$, it is also $\neg\neg$-stable.□

Lemma 1.4.5 (using ??, ??, ??) Let X be a type, let $U \subseteq X$ be an open subtype and let $D \subseteq X$ be dense. Then $U \cap D$ is a dense subtype of U.

Proof Let $V \subseteq U$ be open with $V \cap (U \cap D) = \emptyset$. Using **??**, $V \subseteq X$ is open and $V \cap D = V \cap (U \cap D) = \emptyset$. So $V = \emptyset$ since D is dense in X. □

Being dense is a local property in the following sense:

Lemma 1.4.6 (using ??, ??, ??) Let X be a type and $U_i \subseteq X$ be open subtypes for $i : I$ such that $\bigcup_{i:I} U_i = X$. Then $A \subseteq X$ is dense, if and only if, $A \cap U_i$ is dense in U_i for every i.

Proof Let all $A \cap U_i$ be dense. To show that A is dense, let $V \subseteq X$ be open and $V \cap A = \emptyset$. Then $\emptyset = V \cap A = \bigcup_{i:I} (V \cap U_i) \cap (A \cap U_i)$, so $(V \cap U_i) \cap (A \cap U_i) = \emptyset$ for all $i : I$. But $V \cap U_i$ is open in U_i, so by assumption, $V \cap U_i = \emptyset$ for all $i : I$. So $V = \bigcup_{i:I} V \cap U_i = \emptyset$ and A is dense.

The other direction follows from lemma 1.4.5. □

We will now characterize dense open subsets of affine schemes.

Definition 1.4.7 Let A be a commutative ring.

(a) An element $r : A$ is *nilregular*, if for all $x : A$, such that rx is nilpotent, x is nilpotent.

(b) A list of elements $r_1, \ldots, r_n : A$ is *jointly nilregular*, if for all $x : A$, such that all $r_i x$ are nilpotent, x is nilpotent.

Proposition 1.4.8 Any regular (**??**) element $r : A$ is nilregular.

Lemma 1.4.9 (using ??, ??, ??) Let $X = \operatorname{Spec} A$ be affine and $U \subseteq X$ open. Then U is dense, if and only if, $U = D(r_1, \ldots, r_n)$, with jointly nilregular $r_1, \ldots, r_n : A$.

Proof Let $U \subseteq X$ be dense and open. By **??**, there are $r_1, \ldots, r_n : A$ such that $U = D(r_1, \ldots, r_n)$. Let $x : A$ such that all $r_i x$ are nilpotent. By **??**, this implies $D(r_i x) = \emptyset$ for all i. Since $D(x) \cap D(r_1, \ldots, r_n) = D(r_1 x) \cup \cdots \cup D(r_n x) = \emptyset$, this implies $D(x) = \emptyset$. Therefore x is nilpotent and the r_i are jointly nilregular.

Now let $U = D(r_1, \ldots, r_n)$ with jointly nilregular $r_1, \ldots, r_n : A$. Without loss of generality, let $V = D(f)$ and $D(f) \cap U = \emptyset$. Then $D(r_1 f) \cup \cdots \cup D(r_n f) = \emptyset$, so $D(r_i f) = \emptyset$ for all i. This means $r_i f$ is nilpotent and therefore, f is nilpotent and $D(f) = \emptyset$. □

Corollary 1.4.10 (using ??, ??, ??) The only dense open subset of $1 = \operatorname{Spec} R$ is 1.

Proof Let $U \subseteq 1$ be dense and open. By lemma 1.4.9, there are jointly nilregular $r_1, \ldots, r_n : R$, such that $U = D(r_1, \ldots, r_n)$. But jointly nilregular entails, that one of the r_i is invertible, so $U = 1$. □

Theorem 1.4.11 (using ??, ??, ??)
Let X be a scheme. An open subtype $U \subseteq X$ is dense, if and only if, there is an open affine cover $U_i = \operatorname{Spec} A_i$ and $U \cap U_i = D(r_{i1}, \ldots, r_{in_i})$ with jointly nilregular $r_{i1}, \ldots, r_{in_i} : A_i$ for all i.

Proof By lemma 1.4.9 and lemma 1.4.6. □

Classicly, one possible definition of a dense subset is that the closure is the whole space. We will see an approximation to that in lemma 1.4.14. There are lots of examples of non-trivial dense subsets. For example, the next section will contain a proof, that any non-empty open subset of $\mathbb{A}^1$ is dense.

We can extend the operation from definition 1.1.1 to schemes:

Definition 1.4.12 (using ??, ??, ??) Let X be a scheme and $C \subseteq X$ a closed subscheme. Then C^n is the closed subscheme of X, defined locally as in definition 1.1.1.

Proof We need the axioms to locally get ideals that generate the closed subscheme. We need to show that the construction can be done locally, but this is the case, since for any open affine U, $(C \cap U)^n \subseteq \neg\neg(C \cap U) \subseteq U$ by **??**. □

Lemma 1.4.13 (using ??, ??, ??) Let $U \subseteq X$ be a dense open subtype of a scheme. For any closed subtype V containing U, there merely is an $n : \mathbb{N}$, such that $V^n = X$.

Proof It is enough to do the construction for an open affine $W = \operatorname{Spec} A$, where $V \cap U = \operatorname{Spec} A/(f_1, \ldots, f_n)$ and $U = D(g_1, \ldots, g_l)$. By theorem 1.4.11 we can assume the $g_1, \ldots, g_l$ are jointly nilregular. For any f_i we know $f_i \cdot g_j$ is nilpotent, since

$$\neg D(f_i g_j) = \neg\{x : W \mid f_i g_i(x) \text{ invertible}\} = \emptyset,$$

since if $f_i g_j(x)$ is invertible, then $g_j(x)$ is invertible, but then, we are in U and $f_i(x)$ has to be zero, which contradicts its invertibility.

By the joint nilregularity of the g_j, f_i is nilpotent, so $f_i^n = 0$ and $V^n = W$. □

In the situation of a clopen subset, we get the classical equality:

Lemma 1.4.14 (using ??, ??, ??) Let $U \subseteq X$ be a dense open and closed subtype of a scheme, then $U = X$

Proof By lemma 1.4.13, $U^n = X$. By lemma 1.1.2, we have

$$X \subseteq U^n \subseteq \neg\neg U = U.$$ □

1.5 Closed dense subtypes

This section is due to Hugo Moeneclaey.

Lemma 1.5.1 For any type X, a closed subtype $C : X \to \text{Prop}$ is dense if and only if:

$$\prod_{x:X} \neg\neg C(x)$$

Proof Assume C a closed subtype of X. If C is dense, as $\neg C$ is open and $\neg C \cap C = \emptyset$, we have that $\neg C = \emptyset$ which is precisely what we want.

Conversely assume that for all $x : X$ we have $\neg\neg C(x)$. Let U be an open subtype of X such that $U \cap C = \emptyset$. Then for any $x : X$ we have $\neg(C(x) \wedge U(x))$ as well as $\neg\neg C(x)$, so that we have $\neg U(x)$. So we have $U = \emptyset$ and C is indeed dense. □

Corollary 1.5.2 The type of closed propositions C such that $\neg\neg C$ classifies closed dense subtypes.

Proposition 1.5.3 (using ??, ??) A closed subscheme $\operatorname{Spec}(A/I)$ of an affine scheme $\operatorname{Spec}(A)$ is dense if and only if I is nilpotent.

Proof Assume $\operatorname{Spec}(A/I) \subset \operatorname{Spec}(A)$ dense. For any $f : I$, we have $\operatorname{Spec}(A/I) \cap D(f) = \emptyset$ and $D(f)$ open so that $D(f) = \emptyset$ and f is nilpotent.

Conversely, let I be a finitely generated nilpotent ideal in A generated by $f_1, \cdots, f_n$. Then for all $x : \operatorname{Spec}(A)$, we have $x \in \operatorname{Spec}(A/I)$ if and only if:

$$f_1(x) = 0 \wedge \cdots \wedge f_n(x) = 0$$

But as $f_1, \cdots, f_n$ are nilpotent we have:

$$\neg\neg(f_1(x) = 0) \wedge \cdots \wedge \neg\neg(f_n(x) = 0)$$

so that:

$$\neg\neg(x \in \operatorname{Spec}(A/I))$$

and $\operatorname{Spec}(A/I)$ is dense by lemma 1.5.1. □

1.6 Irreducible and reducible types

We start with the notion of reducible types and will then pass to the negation of this concept, to irreducible types.

Definition 1.6.1 A type is called *reducible*, if there are two disjoint, inhabited open subtypes.

Proposition 1.6.2 The scheme $\operatorname{Spec} R[X,Y]/(XY)$ is reducible.

Proof We take the subsets $D(X), D(Y) \subseteq \operatorname{Spec} R[X,Y]/(XY)$. Then

$$D(X) \cap D(Y) = \{\, (x,y) \mid xy = 0 \wedge x \neq 0 \wedge y \neq 0 \,\} = \emptyset.$$

And $(1,0) \in D(X)$, $(0,1) \in D(Y)$. □

Definition 1.6.3 A type X is called *irreducible*, if the following equivalent propositions hold:

(i) X is not reducible.

(ii) Any non-empty open $U \subseteq X$ is dense.

(iii) For any open disjoint $U, V \subseteq X$ such that $U \neq \emptyset$, we have $V = \emptyset$.

Proposition 1.6.4 Being irreducible is $\neg\neg$-stable.

Proof By the definition as not reducible, or by proposition 1.4.4. □

Example 1.6.5 Every proposition is an irreducible type, since any two inhabited subtypes intersect.

Proposition 1.6.6 (using ??, ??, ??) $\mathbb{A}^1$ is irreducible.

Proof Let $U \subseteq \mathbb{A}^1$ with $U \neq \emptyset$. We have to show that U is dense. Let $U = D(f_1, \dots, f_n)$. We merely have a bound for the degree of each of the $f_i : R[X]$, so we can concatenate all coefficients of the f_i and, since $U \neq \emptyset$, we know that vector is not the zero-vector. So one of the f_i is nilregular by lemma 1.4.1. In particular, the elements $f_1, \dots, f_n$ are jointly nilregular, so U is dense by lemma 1.4.9. □

Example 1.6.7 The scheme $\operatorname{Spec} R[X,Y]/(XY)$ is not irreducible.

Remark 1.6.8 In a classical setting, reducibility and irreducibility are usually defined in terms of closed subsets instead of open subsets. However, this does not give the correct notion in our setting, as the example $\operatorname{Spec} R[X,Y]/(XY)$ shows: this scheme is not the union of the closed subsets $V(X)$ and $V(Y)$.

We will now explore the relation of connectedness and irreducibility. It is not the case, that any open dense subtype of a connected scheme is connected:

Example 1.6.9 Let us first show, that $V(XY) \subseteq \mathbb{A}^2$ is connected. Let $f : V(XY) \to \mathrm{Bool}$ be a function and assume without loss of generality, that $f(0,0) = 1$. Then the restriction of $D(f)$ to both, $V(X)$ or $V(Y)$ is dense. Since $f(x) = 1$ is closed and holds for $x : D(f)$, $f(x) = 1$ holds not not for all $x : V(XY)$, which is enough.

The open subtype $D(X,Y) \subseteq V(XY) \subseteq \mathbb{A}^2$ is not connected. This is witnessed by the function

$$\frac{X}{X+Y}.$$

Proposition 1.6.10 (using no axioms) Any irreducible pointed type is connected.

Proof Let X be an irreducible pointed type and let a decomposition into detachable subsets $X = U \sqcup V$ be given. In particular, U and V are open subsets, and we can assume that the base point of X lies in U. But then U is dense since X is irreducible, so we have $V = \emptyset$ and $U = X$. □

Proposition 1.6.11 (using ??, ??, ??) Let X be an irreducible type and $U \subseteq X$ an open subtype. Then U is also irreducible.

Proof Let $V, W \subseteq U$ be open subtypes with $V \cap W = \emptyset$. Assume that both V and W are nonempty, now we have to show a contradiction. By **??**, V and W are also open subsets of X, so we indeed get a contradiction from the fact that X is irreducible. □

Lemma 1.6.12 Let X be irreducible and $Y : X \to \mathrm{Sch}_{\mathrm{qc}}$ be a family of irreducible types. Then $(x : X) \times Y_x$ is irreducible.

The original version of the following proof is due to David Wärn.

Proof Let $U, V \subseteq X \times Y$ be disjoint open subsets with $(a,b) \in U$, $(c,d) \in V$. Consider the subtypes

$$U_a :\equiv \{\, x : X \mid (x,b) \in U \,\}$$

and

$$V_c :\equiv \{\, x : X \mid (x,d) \in V \,\}.$$

These are two inhabited open subtypes of X, so they can not be disjoint. Since we want to show a contradiction, we can assume we have $e \in U_a \cap V_c$. But then

$$U_e :\equiv \{\, y : Y_e \mid (e,y) \in U \,\}$$

and

$$V_e :\equiv \{\, y : Y_e \mid (e,y) \in V \,\}$$

are open subtypes of Y_e which are disjoint since U and V are disjoint, and inhabited by b respectively d. This contradicts the irreducibility of Y_e. ☐

Lemma 1.6.13 Let X be irreducible and $f : X \to Y$ be surjective, then Y is irreducible.

Proof Using the definition with disjoint opens. ☐

We will see in proposition 3.1.2, that the projective n-space $\mathbb{P}^n$ is irreducible.

1.7 Separated types and apartness

Proposition 1.7.3 was found and proven together with Marc Nieper-Wißkirchen and Ingo Blechschmidt.

Definition 1.7.1 A type X is *separated*, if for all $x, y : X$ the type $x = y$ is a closed proposition, that is, the diagonal $X \to X \times X$ is the embedding of a closed subtype.

Definition 1.7.2 An *apartness relation* on X is a relation $\# : X \to X \to \mathrm{Prop}$, such that it is

(i) irreflexive: $\prod_{x:X} \neg(x \# x)$

(ii) symmetric: $\prod_{x,y:X} x \# y \to y \# x$

(iii) and cotransitive: $\prod_{x,y,z:X} x \# z \to x \# y \vee y \# z$.

Proposition 1.7.3 If X is a separated scheme, then inequality is an apartness relation.

Proof MISSING ☐

Proposition 1.7.4 (using ??, ??, ??) Let X be a separated scheme and U, V be open affine in X. Then $U \cap V$ is affine.

Proof $U \cap V$ is equivalently the closed subtype $\{(x,y) : U \times V \mid x = y\}$. $U \times V$ is affine by **??** and a closed subtype of an affine scheme is affine by **??**. ☐

Example 1.7.5 (using ??, ??) It is not the case that for every finitely presented R-algebra A and every A-module M the map η_M is injective.

Proof Instead of giving a single counterexample, we construct a family of potential counterexamples, indexed by an element $f : R$. We set $A :\equiv R/(f)$ and

$$M :\equiv A^1/\langle\{1 \mid f =_R 0\}\rangle.$$

Then we have $M \otimes x = 0$ for all $x : \mathrm{Spec}\, A$: an element $x : \mathrm{Spec}\, A$ is a witness that $f : R$ is invertible and if f is invertible then $A = 0$, so $M = 0$, so $M \otimes x = 0$. This implies that if η_M is injective then $M = 0$. But we have $M = 0$ if and only if $1_A \in \langle\{1 \mid f = 0\}\rangle$ if and only if 1_A a linear combination (of some length n) of elements of the set $\{1 \mid f = 0\}$ if and only if $f = 0$ $(n > 0)$ or $1 =_A 0$, that is, f is invertible $(n = 0)$. In summary, if η_M is injective for every choice of $f : R$, then every $f : R$ is zero or invertible. But this would be a contradiction to corollary 1.2.4. ☐

2 Reduced schemes

2.1 Reduced schemes (obsolete?)

(What follows is a not completely satisfactory candidate definition of reduced schemes. Marc Nieper-Wißkirchen and Fabian Endres were involved in finding this notion.)

There is a *candidate* definition of reduced schemes. The analogue to the classical definition, that an affine scheme is reduced, if its algebra of functions is reduced, is expected to be useless in the synthetic setup. We start with a notion which is only suitable for affine schemes[1].

Definition 2.1.1 An affine scheme $X = \operatorname{Spec} A$ is *reduced*, if for all functions $f : A$, nilpotency implies $\neg\neg(f = 0)$.

An alternative, stronger criterion would be that if $f : A$ is nilpotent, then $f = r_1 a_1 + \ldots + r_n a_n$ with $r_i : R$ nilpotent and $a_i : A$.

Example 2.1.2 (a) $\mathbb{D}(1)$ is not reduced. The algebra of functions is $R + \varepsilon R$ and we know that ε is nilpotent and non-zero.

(b) $\mathbb{A}^1$ is reduced. To see this, let $f : R[X]$ be nilpotent. Then all coefficients of f are nilpotent and since we proof a double-negation, we can assume they are zero.

(c) A basic open $D(f)$ of an affine reduced scheme is reduced: If $\left(\frac{a}{f^l}\right)^n = 0$, we want to show $\neg\neg\frac{a}{f^l} = 0$. Since we want to show a double negated proposition, we can decide if f is regular or nilpontent. If it is regular, $f^k a^n = 0$ implies $\neg\neg a = 0$ and if it is nilpotent, every function on $D(f)$ is 0 anyway.

(d) Any closed dense proposition, i.e. affine scheme of the form $\operatorname{Spec}(R/(\varepsilon_1, \ldots, \varepsilon_n))$ is reduced.

(e) The cross with one infinitesimal axis $\operatorname{Spec} R[X, Y]/(XY, Y^2)$ is not reduced, since Y is a nilpotent, non-zero function.

From a notion of "reduced" for general schemes, we would expect that it is closed under

(i) taking open subtypes

(ii) and finite open unions.

For the notion of affine reduced scheme from above, (i) holds for basic opens of affines, which is enough to make the following definition well-defined and fullfil the requirements above:

Definition 2.1.3 A scheme X is *reduced* if there is a finite affine open cover $X = \bigcup_i U_i$ such that each U_i is reduced.

In fact, if there is one cover, any cover will be reduced.

Remark 2.1.4 For a reduced scheme X we have:

(i) Any open $U \subseteq X$ is reduced.

(ii) For any finite open affine cover of $X = \bigcup U_i$, all U_i are reduced.

Proof Any open subscheme of an affine scheme is covered by basic opens, so it is reduced. For any cover $(U_i)_i$ of X and a given reduced cover $(V_j)_j$, we have $U_i = \bigcup_j V_j \cap U_i$, so U_i is reduced. □

Example 2.1.5 $V(X^2) \subseteq \mathbb{P}^2$ is not reduced.

2.2 Reduced types

For any type X, we write $\overline{X}$ for the formally étale replacement of X.

Definition 2.2.1 A type X is reduced if for all $P : X \to \mathrm{Prop}$ we have that:

$$(\forall (x : X).\ \overline{P(x)}) \to \overline{\forall (x : X).\ P(x)}$$

Lemma 2.2.2 Reduced types are closed under finite colimits.

[1] An example where this fails for general schemes, is $V(X^2) \subseteq \mathbb{P}^2$.

Proof Because the formally étale replacement commutes with finite limits. ☐

Lemma 2.2.3 Reduced types are closed under Σ-types.

Proof Given X reduced and $(Y_x)_{x:X}$ a family of reduced type, we have that:

$$\begin{aligned}
&\forall(z : (x : X) \times Y_x).\ \overline{P(z)} \\
&= \forall(x : X)(y : Y_x).\ \overline{P(x, y)} \\
&\to \forall(x : X).\ \overline{\forall(y : Y_x).\ P(x, y)} \\
&\to \overline{\forall(x : X)(y : Y_x).\ P(x, y)} \\
&= \overline{\forall(z : (x : X) \times Y_x).\ P(z)}
\end{aligned}$$

☐

Lemma 2.2.4 If we have a surjection $f : X \to Y$ and X is reduced, then Y is reduced.

Proof Assume given $P : Y \to \mathrm{Prop}$. We have that:

$$\begin{aligned}
&\forall(y : Y).\ \overline{P(y)} \\
&= \forall(x : X).\ \overline{P(f(x))} \\
&\to \overline{\forall(x : X).\ P(f(x))} \\
&= \overline{\forall(y : Y).\ P(y)}
\end{aligned}$$

☐

Lemma 2.2.5 Any connected type is reduced.

Proof Let X be connected. Since we want to prove a proposition, we can assume $x : X$. Then we have a surjection:

$$x : 1 \to X$$

and 1 is reduced so we conclude by lemma 2.2.4. ☐

Lemma 2.2.6 A type X is reduced if and only if $\|X\|_0$ is reduced.

Proof If X is reduced, so is $\|X\|_0$ by lemma 2.2.4. Conversly if $\|X\|_0$ is reduced, we have that:

$$X = \sum_{x:\|X\|_0} \sum_{y:X} [y] = x$$

but $\sum_{y:X}[y] = x$ is connected, therefore it is reduced by lemma 2.2.5, so by lemma 2.2.3 we have that X is reduced. ☐

Remark 2.2.7 The stability results for being formally smooth and being reduced are very similar.

2.3 Reduced affine schemes

Lemma 2.3.1 Assume A an f.p. algebra such that for all $a : A$ nilpotent we have that $\overline{a = 0}$. Then for any $f : A$ and $b : A_f$ nilpotent, we have that $\overline{b = 0}$.

Proof We have that $b = \frac{c}{f^k}$ and there exists $n : \mathbb{N}$ such that $b^n = 0$ in A_f, therefore there exists $k : \mathbb{N}$ such that $f^k c^n = 0$ in A. From this we know that fc is nilpotent in A, therefore $\overline{fc = 0}$ and this implies $\overline{b = 0}$ as $fc = 0$ implies $b = 0$. ☐

Lemma 2.3.2 Assume A an f.p. algebra such that for all $a : A$ is nilpotent we have that $\overline{a = 0}$. Then for any Zariski cover:

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

by basic opens and any closed dense $Q \subset \mathrm{Spec}(B)$ we have $\overline{\forall(y : \mathrm{Spec}(B)).\ Q(y)}$.

Proof It is enough to prove the property when $B = A_f$ for $f : A$. Given a closed dense $Q \subset \mathrm{Spec}(A_f)$ we know that Q is of the form $V(b_1, \cdots, b_n)$ for $b_1, \cdots, b_n$ nilpotent in A_f. Then by lemma 2.3.1 we have that $\overline{b_i = 0}$ for all i, therefore:

$$\overline{b_1 = 0 \wedge \cdots \wedge b_n = 0}$$

which means $\overline{Q = \mathrm{Spec}(A_f)}$. □

Lemma 2.3.3 Assume A an f.p. algebra such that for all $a : A$, if a is nilpotent then $\overline{a = 0}$. Then $\mathrm{Spec}(A)$ is reduced.

Proof Assume:

$$P : \mathrm{Spec}(A) \to \mathrm{Prop}$$

such that:

$$\forall (x : \mathrm{Spec}(A)).\ \overline{P(x)}$$

This means that:

$$\forall (x : \mathrm{Spec}(A)). \exists (Q : \text{closed dense}).\ Q \to P(x)$$

By Zariski local choice we get a Zariski cover by basic opens:

$$f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

and a family of closed dense propositions $Q : \mathrm{Spec}(B) \to \mathrm{Prop}$ such that:

$$\forall (y : \mathrm{Spec}(B)).\ Q(y) \to P(f(y))$$

By lemma 2.3.2 we have that:

$$\overline{\forall (y : \mathrm{Spec}(B)).\ Q(y)}$$

so we have that:

$$\overline{\forall (y : \mathrm{Spec}(B)).\ P(f(y))}$$

which is equivalent to:

$$\overline{\forall (x : \mathrm{Spec}(A)).\ P(x)}$$

by the surjectivity of f. □

Next lemma is just generic business with modalities, and should probably be moved elsewhere.

Lemma 2.3.4 Assume given Y formally étale and $f : X \to Y$ formally étale-surjective, meaning that for all $y : Y$ we have:

$$\overline{\|\mathrm{fib}_f(y)\|}$$

Then the induced map:

$$f' : \overline{X} \to Y$$

is surjective.

Proof Assume $y : Y$, we want to prove:

$$\exists (x : \overline{X}).\ f'(x) =_Y y$$

but this type is a formally étale proposition as it is the truncation of a formally étale type.

Therefore when proving it we can assume $x : \mathrm{fib}_f(y)$ as by hypothesis $\overline{\|\mathrm{fib}_f(y)\|}$, and $[x]$ gives a witness. □

Next proposition could be called the reduced duality.

Proposition 2.3.5 Let A be an f.p. algebra such that $\mathrm{Spec}(A)$ is reduced. Then the map:

$$\overline{A} \to \overline{R}^{\mathrm{Spec}(A)}$$

is an equivalence.

Proof First we prove that the map is injective, and then surjective.

- In order to prove injectivity, it is enough to prove that for all $a : A$ such that:
$$\forall (x : \mathrm{Spec}(A)).\ [a(x)] =_{\overline{R}} [0]$$
we have that:
$$[a] =_{\overline{A}} [0]$$
But this just means that:
$$\forall (x : \mathrm{Spec}(A)).\ \overline{a(x) = 0}$$
implies that:
$$\overline{a = 0}$$
which is a clear consequence of $\mathrm{Spec}(A)$ being reduced.

- For surjectivity, by lemma 2.3.4 it is enough to prove that the map:
$$A \to \overline{R}^{\mathrm{Spec}(A)}$$
is formally-étale surjective. So it means than given a map $f : \mathrm{Spec}(A) \to \overline{R}$ we need to merely find a dotted lift in:
$$\begin{array}{ccc} & & R \\ & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \overline{R} \end{array}$$
up to formally étale replacement. By Zariski local choice we get local lifts g_i on a cover of by basic opens $(U_i)_{i:I}$. Consider $i, j : I$, then for all $x : U_i \cap U_j$ we have that $[g_i(x)] =_{\overline{R}} [g_j(x)]$ which means that $g_i - g_j$ is nilpotent in $U_i \cap U_j$. So by lemma 2.3.1 we have that:
$$\overline{g_i = g_j}$$
when restricted to the basic open $U_i \cap U_j$. But there is finitely many such i, j, so when building the lift up to formally étale replacement we can assume $g_i = g_j$ for all i, j and get a global lift. □

Remark 2.3.6 For A f.p. we have that:
$$\overline{\mathrm{Spec}(A)} = \mathrm{Hom}_R(A, \overline{R}) = \mathrm{Hom}_{\overline{R}}(\overline{A}, \overline{R})$$
where the first step relies crucially on A being f.p. and formally étale being lex, and the second step working for any lex modality. (NOT CHECKED IN DETAILS)

Theorem 2.3.7
Let A be an f.p. algebra. The following are equivalent:
(i) $\mathrm{Spec}(A)$ is reduced.

(ii) For all $a : A$ nilpotent, we have that $\overline{a = 0}$.

(iii) For all $a : A$ nilpotent, there are $r_1, \cdots, r_n : R$ nilpotent and $a_1, \cdots, a_n : A$ such that:
$$a = r_1 a_1 + \cdots + r_n a_n$$

(iv) The map:
$$\overline{A} \to \overline{R}^{\mathrm{Spec}(A)}$$
is an equivalence.

Proof It goes as follow:
- (iii) implies (ii). When proving:
$$\overline{r_1 a_1 + \cdots + r_n a_n = 0}$$
we can assume $r_i = 0$ for all i, as they are nilpotent.

- (ii) implies (iii). Assume $a : A$ such that $\overline{a = 0}$, then there is an f.g. nilpotent ideal I in R such that:

$$I = 0 \to a = 0$$

then the image of a under:

$$A \to A^{I=0}$$

is 0, and since A is strongly quasi-coherent we have that:

$$A^{I=0} = A \otimes R/I = A/IA$$

so $a \in IA$ and we can conclude.

- (ii) implies (i). By lemma 2.3.3.
- (i) implies (iv). By proposition 2.3.5.
- (iv) implies (ii). Assume $a : A$ nilpotent, then for all $x : \mathrm{Spec}(A)$ we have $a(x)$ nilpotent so that:

$$[a(x)] =_{\overline{R}} [0]$$

So the map:

$$A \to \overline{R}^{\mathrm{Spec}(A)}$$

sends a to 0. By the assumed duality, we have that:

$$[a] =_{\overline{A}} [0]$$

which precisely means:

$$\overline{a = 0}$$

as being formally étale is a lex modality. □

Remark 2.3.8 We did not manage to prove that it is not equivalent to definition 2.1.1. Is it?

2.4 Examples

For affine schemes we always use criteria (ii) from theorem 2.3.7.

Lemma 2.4.1 For all $n : \mathbb{N}$, we have that R^n is reduced.

Proof If $P : R[X_1, \cdots, X_n]$ is nilpotent then all its coefficients are nilpotent, therefore when proving $\overline{P = 0}$ we can assume all the coefficients of P are 0. □

Lemma 2.4.2 Open propositions are reduced.

Proof By lemma 2.3.1 this holds for basic open proposition. We conclude by lemma 2.2.2 and lemma 2.2.4.□

Corollary 2.4.3 If $(U_i)_{i:I}$ is a Zariski cover of X, then X is reduced if and only if U_i is reduced for all $i : I$.

Lemma 2.4.4 Closed dense propositions are reduced.

Proof Assume I nilpotent f.g. ideal in R, and $s : R/I$ such that $s^n = 0$ in R/I. There we can get an $r : R$ such that $[r] = s$ in R/I and $r^n \in I$. Then r is nilpotent, so $\overline{r = 0}$ and this implies $\overline{s = 0}$. □

Lemma 2.4.5 The affine scheme $\mathrm{Spec}(R[X,Y]/(XY))$ is reduced.

Proof We have $A :\equiv R \cdot 1 \oplus XR[X] \oplus YR[Y] = R[X,Y]/(XY)$ as R-modules, where the multiplication in A is given by multiplication of polynomials, forgetting all coefficients of mixed monomials. For a general nilpotent $\alpha + P + Q$ in A with $\alpha : R$, $P : XR[X]$ and $Q : YR[Y]$:

$$0 = (\alpha + P + Q)^n = \sum_{k=0}^{n} \binom{n}{n-k} \alpha^{n-k}(P^k + Q^k)$$

means that α is nilpotent. But then all terms in the sum to the right are nilpotent, except $(P^n + Q^n)$. But then $(P^n + Q^n)$ is nilpotent as well, which means that P and Q are nilpotent polynomials. So we can conclude with Theorem 2.3.7 (iii) . □

We expect that standard étale scheme are reduced. We have not proven it yet, but we have the following:

Remark 2.4.6 If standard étale schemes are reduced, then any smooth scheme is reduced.

3 Applications

3.1 Topological Properties of Projective Space

Proposition 3.1.1 $\mathbb{P}^n$ is separated.

Proof We have to show that $x = y$ is closed for all $x, y : \mathbb{P}^n$. Since we are proving a proposition, we may assume represantatives $[x_0 : \cdots : x_n] = [y_0 : \cdots : y_n]$ and an index i such that x_i is invertible. Let $\lambda :\equiv \frac{y_i}{x_i}$, then $x = y$ is equivalent to

$$\prod_j \lambda x_j = y_j$$

– which is closed. □

Proposition 3.1.2 (using ??, ??, ??) $\mathbb{P}^n$ is irreducible.

Proof By proposition 1.6.6 and lemma 1.6.12, $\mathbb{A}^{n+1}$ is irreducible. $\mathbb{A}^{n+1} \setminus \{0\}$ is an open subtype of $\mathbb{A}^{n+1}$, so it is also irreducible by proposition 1.6.11. Finally, the projection $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is surjective, so by lemma 1.6.13, $\mathbb{P}^n$ is irreducible. □

The following conclusion could also be drawn from our results about functions on $\mathbb{P}^n$ in the next section.

Corollary 3.1.3 (using ??, ??, ??) $\mathbb{P}^n$ is connected.

Proof Note first that $\mathbb{P}^n$ is pointed by $[1 : 0 : \cdots : 0]$. By proposition 3.1.2, $\mathbb{P}^n$ is irreducible and by proposition 1.6.10 any irreducible pointed type is connected. □

3.2 A Property of R

Theorem 3.2.1 (using ??, ??, ??)
The ring R is not coherent, i.e. it is not the case, that all finitely generated ideals in R are finitely presented.

Proof We will show, that it is not the case, that any R-module map $R \to R$ has a finitely generated kernel. Every R-linear map $\varphi : R \to R$ is of the form $\varphi_x(z) = xz$ for some $x : R$, namely $x :\equiv \varphi(1)$. Assume it is always possible to find generators $y_1, \ldots, y_n : R$ of the kernel of φ_x. That means there is a map

$$c : \prod_{x:R} \left(\exists_{y_1,\ldots,y_n:R} \prod_{z:R} \left(\varphi_x(z) = 0 \text{ iff } \exists_{\lambda_1,\ldots,\lambda_n:R} z = \sum_{i=1}^{n} \lambda_i y_i \right) \right).$$

By **??** and boundedness (**??**), we translate the first "$\exists$" to a function $g : D(f) \to R^n$ on a neighborhood $D(f) \subseteq R$ of $0 : R$. We know that if x is invertible, then $\ker(\varphi_x) = (0)$, which means $y_i = 0$ for all i. So $g(x)$ must be the 0-vector for all $x : D(f) \cap D(X)$. Since $\mathbb{A}^1$ is irreducible by proposition 1.6.6, $D(X)$ and $D(f)$ are both already dense by being non-empty. By lemma 1.4.5 $D(f) \cap D(X)$ is dense in $D(f)$, so by lemma 1.4.13 applied to $V(g_i) \subseteq D(f)$, the entries $g_i(x)$ of $g(x)$ must be nilpotent for all $x : D(f)$. But this is a contradiction, since for $x = 0$, the kernel of φ_x is R and there must be an invertible entry in $g(x)$. □

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