Stacks in Synthetic Algebraic Geometry

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Abstract

The aim is to define and reason about Deligne-Mumford stacks. Authors so far: Felix Cherubini, Hugo Moeneclaey.

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Contents

1 Etale and smooth atlas

1.1 Definition

Remark 1.1.1 Here we only consider étale atlas. This should probably be extended to smooth and flat atlas.

Definition 1.1.2 A type X has an étale atlas if there merely exists an affine scheme U with a formally étale surjection:

$$U \to X$$

Remark 1.1.3 Any scheme has an étale atlas, so we could replace affine scheme by scheme in the definition of étale atlas.

Definition 1.1.4 A map $f : X \to Y$ has an étale atlas if X and Y have étale atlas U and V with commutative square:

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{f}} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

We say that $\widetilde{f}$ is an étale atlas for f.

Lemma 1.1.5 Let X, Y be types with étale atlas, then any $f : X \to Y$ has an étale atlas.

Proof We merely have a formally étale surjections $v : V \to Y$ and $u : U \to X$ by assumption. Let P' be the iterated pullback:

$$\begin{array}{ccccc} P' & \longrightarrow & P & \longrightarrow & V \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle v} \\ U & \xrightarrow[u]{} & X & \xrightarrow[f]{} & Y \end{array}$$

Then $P = (x : X) \times W_x$, with $W_x :\equiv (z : V) \times f(x) = v(z)$. As a dependent sum of schemes, W_x is a scheme. For the same reason, $P' = (x : U) \times W_{u(x)}$ is a scheme. As iterated pullback of a formally étale surjection, p' is a formally étale surjection and therefore also the composition with u. □

1.2 Basic properties

Lemma 1.2.1 Let X be a type with an atlas. Any function from X to $\mathbb{N}$ is merely bounded.

Proof This holds for schemes. So given a surjective map $U \to X$ with U a scheme, this holds for X. □

Lemma 1.2.2 Let X be a type with a formally étale (resp. formally smooth) atlas. Then X has étale-local (resp. smooth-local) choice.

Proof Immediate from the definition using the fact that Zariski-local choice implies étale-local (resp. smooth-local) choice. □

1.3 Étale-local properties of types and morphisms

Definition 1.3.1 A property P of types is étale-local if $P(1)$ holds, P is stable by dependent sum and given an étale surjection:

$$X \to Y$$

we have $P(X)$ if and only if $P(Y)$.

Definition 1.3.2 A property P of morphisms between types is called étale-local if:

- We have $P(\mathrm{id})$, and P is stable by composition.
- Given a pullback square:

  $$\begin{array}{ccc} U & \longrightarrow & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & Y \end{array}$$

  where the bottom map is a formally étale surjection, we have that $P(f)$ if and only if $P(g)$.
- Given a commutative triangle:

  $$\begin{array}{ccc} U & \longrightarrow & V \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y & \end{array}$$

  where the top map is a formally étale surjection, we have that $P(f)$ if and only if $P(g)$.

Any étale local class of maps contains all formally étale surjection. The conjunction of étale-local property is étale-local.

Lemma 1.3.3 Assume given an étale-local property of types P. Then having fibers in P is an étale-local property of morphisms.

Proof Since $P(1)$ we know that fibers of equivalences obey P. Stability of P by dependent sums gives stability of maps with fibers satisfying P by composition.

Given a pullback square:

$$\begin{array}{ccc} U & \longrightarrow & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & Y \end{array}$$

with the bottom map surjective, we know that fibers of f have property P if and only if fibers of g do.

Given a commutative triangle:

$$\begin{array}{ccc} U & \longrightarrow & V \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y & \end{array}$$

with the top map étale surjective, we know that for any $y : Y$ there is a formally étale surjection:

$$\mathrm{fib}_g(y) \to \mathrm{fib}_f(y)$$

so we can conclude using étale-locality of P. □

Lemma 1.3.4 The following propriety of morphisms are étale-local:

(i) Being surjective.

(ii) Being formally smooth.

(iii) Being formally étale.

(iv) Being surjective and formally smooth.

(v) Being surjective and formally étale.

Proof Just need to prove (i)-(iii) as conjunction of étale-local propriety are étale-local.

By lemma 1.3.3, we just need being merely inhabited, formally smooth and formally étale are étale-local property of types. Stability by unit and dependent sum is straightforward.

(i) If we have a formally étale surjection:

$$X \to Y$$

then X is merely inhabited if and only if Y is.

(ii) If we have a formally étale surjection:

$$X \to Y$$

then if Y is formally smooth so is X as the map $X \to Y$ is formally smooth. Conversely if X is formally smooth then by **??** so is Y as the map $X \to Y$ is surjective.

(iii) If we have a formally étale surjection:

$$X \to Y$$

then if Y is formally étale so is X as the map $X \to Y$ is formally étale. Conversely assume X formally étale. Then the square:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{Et}(X) & \longrightarrow & \mathrm{Et}(Y) \end{array}$$

is a pullback square as the top map is formally étale and being formally étale is a lex modality. Since X is formally smooth so is Y by (ii), so that the right map of the square is surjective. Then the bottom map is surjective and since the left map is an equivalence, we can conclude that right map is an equivalence and Y is formally étale. □

Remark 1.3.5 We expect flat and open maps to be étale-local as well, this time not using a fiberwise characterisation.

1.4 Morphisms and atlas

Now we check the maps with an étale-local propriety have scheme cover enjoying the same property.

Lemma 1.4.1 Assume given a morphism f between types with étale atlas and an étale-local property P of morphisms. The following are equivalent:

(i) The map f has property P.

(ii) There exists an atlas for f that has property P.

Proof Assume given a map $f : X \to Y$ such that $P(f)$ for P an étale-local propriety. Then as in the previous lemma we consider the iterated pullbacks:

$$\begin{array}{ccccc} P' & \xrightarrow{u'} & P & \xrightarrow{g} & V \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle v} \\ U & \xrightarrow[u]{} & X & \xrightarrow[f]{} & Y \end{array}$$

Since $P(f)$ we have $P(g)$ as g is a pullback of f along a formally étale surjective maps. Then u formally étale surjective implies u' formally etale surjective and then $P(g \circ u')$, giving a cover of f having P.

Conversely assume given a commutative square:

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y \end{array}$$

with vertical maps formally étale surjective, such that $P(g)$. Then we know that $P(v)$ holds as v is formally étale surjective, so that we have $P(v \circ g)$ as P is stable by composition, and then $P(f \circ u)$ with u formally étale surjective implies $P(f)$. □

Corollary 1.4.2 A map between types with is surjective (resp. formally smooth, formally étale, formally smooth surjective, formaly étale surjective) if and only if it has a surjective (resp. formally smooth, formally étale, formally smooth surjective, formally étale surjective) atlas.

Proof From lemma 1.4.1 and lemma 1.3.4. □

2 Algebraic ∞-stacks

2.1 Definition and basic properties

Definition 2.1.1 A Deligne-Mumford ∞-stack is a type X such that:

- The type X merely has a formally étale atlas.
- Coinductively, identity types in X are Deligne-Mumford ∞-stacks.

Deligne-Mumford ∞-stacks enjoys boundedness of map to $\mathbb{N}$, as well as étale-local choice.

Definition 2.1.2 An Artin ∞-stack is a type X such that:

- The type X merely has a formally smooth atlas.
- Identity types in X are Deligne-Mumford ∞-stacks.

Artin ∞-stacks enjoys boundedness of map to $\mathbb{N}$, as well as smooth-local choice.

Remark 2.1.3 In the previous definition one would be typically ask that X is an fppf or étale sheaf, and the atlas would only be required to be fppf or étale surjective rather than surjective. We leave it as is for now.

Definition 2.1.4 A Deligne-Mumford (resp. Artin) ∞-stack that is an n-type is called a Deligne-Mumford (resp. Artin) n-stack.

Remark 2.1.5 Our definition is unusual in several ways:

- We consider stacks without any truncation hypothesis, whereas traditionally one only consider Deligne-Mumford 0-stacks (called algebraic spaces), Artin 1-stacks (often called algebraic stacks) and Deligne-Mumford 1-stacks (called Deligne-Mumford stacks). While it is unclear to us if considering k-stacks for $k > 1$ is useful, setting things up this way allows us to factor proofs of results for 0 and 1-stacks.
- Traditionally, identity types in algebraic spaces are assumed to be schemes that are propositions, whereas we just assume Deligne-Mumford ∞-stacks. We will see in **??** that our hypothesis is less restrictive. We could mimic this by asking for a -1-stack to be a propositional scheme, and define inductively $n+1$-stacks as a type with an appropriate cover and n-stacks as identity types.

  Our set up allow for smoother generalisation of properties, e.g. it is obvious for us that an $n+1$-stack that is an n-type is an n-stack, whereas this fails for $n = -1$ using the traditional definition)

We might change our mind in the future about how to set things up.

2.2 Basic stability results

Here we prove that schemes are ∞-stacks and that ∞-stacks are stable by finite limits and the appropriate notion of quotient.

Remark 2.2.1 Contractible types are Deligne-Mumford ∞-stacks.

Lemma 2.2.2 Schemes are Deligne-Mumford ∞-stacks.

Proof Given a scheme X its Zariski cover by affine schemes gives an étale atlas for X, using the fact that finite sums of open propositions are formally étale. Then we know that identity types in schemes are schemes and we can conclude coinductively. □

Lemma 2.2.3 Assume given X a type with Y_x a type depending on $x : X$. If we have a formally étale (resp. formally smooth) surjection:

$$p : U \to X$$

and for all $u : U$ a formally étale (resp. formally smooth) surjection:

$$q_x : V_u \to Y_{p(u)}$$

then the induced map in:

$$\sum_{u:U} V_u \to \sum_{x:X} Y_x$$

is a formally étale (resp. formally smooth) surjection.

Proof Formally étale (resp. formally smooth) merely inhabited types are closed under dependent sums, and fibers of the induced maps are dependent sums of fibers of p and some q_x, so they are formally étale (resp. formally smooth) and merely inhabited. □

Next result implies that Deligne-Mumford and Artin ∞-stacks are stable under finite limits, as it is obvious that they are stable under identity types.

Proposition 2.2.4 Deligne-Mumford (resp. Artin) ∞-stacks are stable by dependent sums.

Proof Assume given a Deligne-Mumford (resp. Artin) ∞-stacks X and for all $x : X$ an algebraic space Y_x. We need to merely find a scheme-cover for $\sum_{x:X} Y_x$.

For any $x : X$ we merely have:

$$\sum_{V_x:\text{Scheme}} \sum_{q_x:V_x\to Y_x} q_x \text{ étale (resp. smooth) surjection}$$

So by étale-local (resp. smooth-local) choice for X, there merely is a scheme U with an formally étale (resp. formally smooth) surjection $p : U \to X$ such that we merely have:

$$\prod_{u:U} \sum_{V_u:\text{Scheme}} \sum_{q_u:V_u\to Y_{p(u)}} q_u \text{ étale (resp. smooth) surjection}$$

Then we merely have a scheme $\sum_{u:U} V_u$ with an induced map:

$$\sum_{u:U} V_u \to \sum_{x:X} Y_x$$

which is formally étale (resp. formally smooth) surjective by lemma 2.2.3.

Then we know that dependent sums commutes with identity so we can conclude coinductively. □

We can even do a bit better:

Lemma 2.2.5 Let us assume a pullback square:

$$\begin{array}{ccc} X & \longleftarrow & X \times_Z Y \\ \downarrow & & \downarrow \\ Z & \longleftarrow & Y \end{array}$$

where Y is an Artin ∞-stack and X and Y and Deligne-Mumford ∞-stack. Then: $X \times_Z Y$ is a Deligne-Mumford ∞-stack.

Next result should be interpreted as saying that Deligne-Mumford and Artin ∞-stacks are stable under nice quotients:

Proposition 2.2.6 Assume given a type Y such that there merely exists:

- A Deligne-Mumford (resp. Artin) ∞-stack X.
- A formally étale (resp. formally smooth) surjective map:

$$f : X \to Y$$

which fibers are Deligne-Mumford ∞-stacks.

Then Y is a Deligne-Mumford (resp. Artin) ∞-stack.

Proof It is clear from the hypothesis that Y has an étale (resp. smooth) atlas. Now we need to prove that identity types in Y are Deligne-Mumford ∞-stack. But for all $x, x' : X$ we have the following pullback square:

$$\begin{array}{ccc} \sum_{x:X} f(x) =_Y f(x') & \longleftarrow & f(x) =_Y f(x') \\ \downarrow & & \downarrow \\ X & \xleftarrow[x]{} & 1 \end{array}$$

Where X is a Deligne-Mumford (resp. Artin) ∞-stack and $\sum_{x:X} f(x) =_Y f(x')$ is a Deligne-Mumford ∞-stack by hypothesis, so that for all $x, x' : X$ we have that $f(x) =_Y f(x')$ is a Deligne-Mumford ∞-stack. We can conclude using the surjectivity of f. □

2.3 ∞-stacks and group actions

Lemma 2.3.1 Let G be a higher groups that is formally étale (resp. formally smooth) and a Deligne-Mumford ∞-stack. Then BG is a Deligne-Mumford (resp. Artin) ∞-satck.

Proof We use proposition 2.2.6 on the map:

$$1 \to BG$$

Indeed the fibers of this maps are all merely equivalent to G, so that they are Deligne-Mumford ∞-stack and formally étale (resp. formally smooth), as well as merely inhabited. □

Corollary 2.3.2 Let G be a higher groups that is formally étale (resp. formally smooth) and a Deligne-Mumford ∞-stack. Assume G acts on a Deligne-Mumford (resp. Artin) ∞-stack X. Then the homotopy quotient $X//G$ is a Deligne-Mumford (resp. Artin) ∞-stack.

Proof By proposition 2.2.4 and lemma 2.3.1. □

2.4 Tiny types

This section might better be placed elsewhere at a latter point.

Definition 2.4.1 A type D is tiny if:

- The type D has choice.
- Given a family $X(d)$ of affine shemes for $d : D$, the type:

  $$\prod_{d:D} X(d)$$

  is an affine scheme.

Remark 2.4.2 For our application in the next section, it is enough to ask that given a family $X(a)$ of affine shemes for $a : A$, the type:

$$\prod_{a:A} X(a)$$

has an étale (resp. smooth) atlas.

Lemma 2.4.3 Finite types are tiny.

Proof Because finite types always have choice, and finite product of affine schemes are affine. □

Lemma 2.4.4 Tiny types are stable by dependent sums.

Proof Both comes straightforwardly from the adjunction between dependent sums and dependent products. □

Lemma 2.4.5 Assume A a finitely presented algebra such that A is merely equivalent to R^n as an R-module. Then given any family of f.p. algebras B_x for $x : \mathrm{Spec}(A)$, the type:

$$\prod_{x:\mathrm{Spec}(A)} \mathrm{Spec}(B_x)$$

is an affine scheme.

Proof We have an f.p. A-algebra B such that:

$$\prod_{x:\mathrm{Spec}(A)} \mathrm{Spec}(B_x)$$

is equivalent to the type of sections of the map:

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

We have that B is an f.p. R-algebra, assume it is of the form:

$$B = R[X_1, \cdots, X_n]/P_1, \cdots, P_m$$

and A is of the form R^k as an R-module.

We consider $Q_1, \cdots, Q_k$ the images of the canonical basis $e_1, \cdots, e_k$ of R^k under the map $A \to B$ making B an A-algebra.

Then a section of the map:

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is equivalent to a map of algebra:

$$\psi : R[X_1, \cdots, X_n]/P_1, \cdots, P_m \to R^k$$

such that:

$$\psi(Q_j) = e_j$$

so it is equivalent to giving:

$$x_1, \cdots, x_n : R^k$$

such that:

$$P_i(x_1, \cdots, x_n) = 0$$

$$Q_j(x_1, \cdots, x_n) = e_j$$

This is an affine scheme. □

Lemma 2.4.6 Any finite sums of infinitesimal variety X such that R^X is merely iso to R^n as an R-module is tiny.

Proof By lemma 2.4.4 it is enough to prove this for one such infinitesimal variety. This holds by lemma 2.4.5 together with the fact that infinitesimal variety have choice. □

Example 2.4.7 Finite sums of standard infinitesimal disks are tiny.

Does $\mathrm{Spec}(R[X]/g)$ has choice for g monic? This would be a good for the fppf topology.

2.5 ∞-stacks and exponentials

Lemma 2.5.1 Assume given a tiny type D. Then given a family $P(d)$ of Deligne-Mumford (resp. Artin) ∞-stack for $d : D$, the type:

$$\prod_{d:D} P(d)$$

is a Deligne-Mumford (resp. Artin) ∞-stack.

Proof First we prove that we have an étale (resp. smooth) atlas for:

$$\prod_{d:D} P(d)$$

Indeed using choice for A, we merely get for all $d : D$ an affine scheme $X(d)$ and a formally étale (resp. formally smooth) map:

$$X(d) \to P(d)$$

The induced map:

$$\prod_{d:D} X(d) \to \prod_{d:D} P(d)$$

is surjective because D has choice, and formally étale using general proprieties of modalities (resp. formally smooth using the fact that D has choice). But:

$$\prod_{d:D} X(d)$$

has an étale (resp. smooth) atlas by hypothesis and therefore so does:

$$\prod_{d:D} P(d)$$

The coinductive step is straightforward using the commutation of identity types and dependent products. □

We can apply this to lemma 2.4.6.

2.6 ∞-stacks and truncations

Proposition 2.6.1 Let X be a formally étale Deligne-Mumford ∞-stack. Then for all n we have that $\|X\|_n$ is a formally étale Deligne-Mumford ∞-stack.

Proof We proceed by induction on n. The base case $n = -2$ is trivial. Assume it holds for $n - 1$. Then consider:

$$[\_] : X \to \|X\|_n$$

To apply proposition 2.2.6 and conclude, it is enough to prove that the fibers of this map are formally étale Deligne-Mumford ∞-stack, but the fiber over $[y] : \|X\|_n$ is:

$$\sum_{x:X} \|x =_X y\|_{n-1}$$

and by induction we know that $\|x =_X y\|_{n-1}$ is a formally étale Deligne-Mumford ∞-stack, so we can conclude since by the surjectivity of $[\_]$ any fiber merely is of this form. □

2.7 A remark on the fundamental theorem for stacks

Any surjective map:

$$f : X \to Y$$

morally means that Y is the quotient of X by a kind of ∞-pregroupoid structure on X defined via:

$$\mathrm{Hom}(x, x') = f(x) =_Y f(x')$$

It is not possible to state this precisely because it would involve infinite towers of coherences that we don't know how to write down in HoTT. But when X is an affine scheme and Y is an n-type for n low enough, it can be worked out it details. We will do this for $n < 2$ in the rest of the notes, under the name of fundamental theorem for n-stacks.

3 Algebraic propositions

3.1 Definition

Definition 3.1.1 A Deligne-Mumford (resp. Artin) proposition is a Deligne-Mumford (resp. Artin) (-1)-stack.

Remark 3.1.2 A proposition is Deligne-Mumford (resp. Artin) if and only is it has an étale (resp. smooth) atlas.

We suspect any Artin propositon is in fact Deligne-Mumford.

3.2 Fundamental theorem for algebraic propositions

Proposition 3.2.1 A proposition is Deligne-Mumford (resp. Artin) if and only if it is merely of the form:

$$\|\mathrm{Spec}(A)\|$$

where $\|\mathrm{Spec}(A)\|$ implies that $\mathrm{Spec}(A)$ is formally étale (resp. formally smooth).

Proof Let P be such a proposition, then we merely have an étale (resp. smooth) atlas:

$$\mathrm{Spec}(A) \to P$$

Since this map is surjective it induces an equivalence:

$$\|\mathrm{Spec}(A)\| \simeq P$$

and the fact that:

$$\mathrm{Spec}(A) \to \|\mathrm{Spec}(A)\|$$

is étale (resp. smooth) is precisely the given condition. The converse if clear. □

3.3 Examples

Lemma 3.3.1 Any propositional scheme is a Deligne-Mumford proposition.

Lemma 3.3.2 Any étale (resp. smooth) scheme X, the type $\|X\|$ is a Deligne-Mumford (resp. Artin) proposition.

Proof Immediate using the Zariski cover of X by an étale (resp. smooth) affine scheme. □

Lemma 3.3.3 Any fomally étale (resp. formally smooth) Deligne-Mumford ∞-stack X, the type $\|X\|$ is a Deligne-Mumford (resp. Artin) proposition.

3.4 Schemes are étale sheaves (obsolete)

Lemma 3.4.1 Assume given A an f.p. algebra, then for any $h_1, \cdots, h_n : A$ such that the open $D(h_1, \cdots, h_n)$ is constant, for any $x : \mathrm{Spec}(A)$ we have that:

$$x \in D(h_1, \cdots, h_n) \leftrightarrow \exists i.h_i \text{ not nilpotent}$$

Proof TODO □

Lemma 3.4.2 Any formally étale type X is reduced, meaning that any closed dense embedding into X not not has a section.

Proof It is immediate that is actually has a section. □

Corollary 3.4.3 For any affine étale scheme $\mathrm{Spec}(A)$we have that:

$$h \text{ nilpotent} \leftrightarrow \neg\neg h = 0$$

Proof The reverse implication is always true. If h is nilpotent then for all $x : \mathrm{Spec}(A)$ is nilpotent, so that:

$$\prod_{x:\mathrm{Spec}(A)} \neg\neg h(x) = 0$$

But then we have that $V(h)$ is a closed dense embedding, so it not not has a section, meaning $\neg\neg h = 0$. □

3.5 Not all Deligne-Mumford propositions are schemes

Lemma 3.5.1 If the map:

$$\psi : \mathbb{A}^1 \to \mathbb{A}^1$$

$$\psi(x) = x^p$$

is surjective on an open U, then this open is empty.

Proof Assume an open $U \subset \mathbb{A}^1$ such that:

$$\psi : \psi^{-1}(U) \to U$$

is surjective. Assume $a \in U$, by Zarsiki-local choice we have $f, g : R[X]$ such that $a \in D(g)$ and we have:

$$\frac{f}{g^n}$$

inverse to ψ. Since $g(a)$ is invertible, we have that g is regular, so that:

$$\frac{f^p}{g^{pn}} = X$$

implies that:

$$f^p = g^{pn} X$$

By induction we prove that all the coefficients of f and g are nilpotent, which contradicts $g(a) \neq 0$. $\square$

Proposition 3.5.2 Not all Deligne-Mumford propositions are schemes.

Proof We have that $p \neq 0$ for some prime p, because locality of the ring implies that $2 \neq 0$ or $3 \neq 0$. Then for all $a : R$ such that $a \neq 0$, we consider the étale affine scheme:

$$E_a = \text{Spec}(R[X]/X^p - a)$$

We see that the propositions $\|E_a\|$ are Deligne-Mumford. We assume that they are schemes and reach a contradiction. Indeed then $\|E_a\|$ would be an fppf sheaf by **??**, so it would be $\|E_a\|$-local, so it would be inhabited. This means that thet map:

$$\psi : R \to R$$

$$\psi(x) = x^p$$

would be surjective on $R^\times$. This contradicts lemma 3.5.1. $\square$

A natural question to ask at this point is wether any Deligne-Mumford proposition that is an fppf sheaf is a scheme?

4 Algebraic spaces

4.1 Definition and basic properties

Definition 4.1.1 An algebraic space is a Deligne-Mumford 0-stack.

Map from an algebraic space to $\mathbb{N}$ are bounded, and algebraic spaces have étale-local choice.

Remark 4.1.2 In the traditional definition it is additionally required that identity types in an algebraic space are schemes. Maybe we will change this later.

4.2 Fundamental theorem of algebraic spaces

In brief, algebraic spaces are quotients of schemes by Deligne-Mumford étale equivalence relations.

Lemma 4.2.1 Assume given a set X then the map:

$$\sum_{R:X\to X\to \mathrm{Prop}} R \text{ equivalence relation} \to \sum_{Y:\mathrm{Set}} \sum_{p:X\to Y} p \text{ surjective}$$

$$R \mapsto (X/R, [\_])$$

is an equivalence with inverse the map:

$$(Y, p) \mapsto \lambda x, y.p(x) = p(y)$$

Proof Plain HoTT, beware that we need to use the set-truncation to define the quotient. ☐

Definition 4.2.2 An equivalence relation R on a type X is called:

- Deligne-Mumford if for all $x, y : X$ the proposition $R(x, y)$ is Deligne-Mumford.
- Étale if for any $x : X$ its fibers:
  $$\sum_{x:X} R(x, y)$$
  are formally étale.

Proposition 4.2.3 Assume given a set X, then the following types are equivalent:

- The type of Deligne-Mumford étale equivalence relation over X.
- The type of sets Y with Deligne-Mumford identity types and a surjective formally étale map from X to Y.

Proof By the equivalence in lemma 4.2.1, it is enough to check that:

- The identity types in X/R are Deligne-Mumford if and only if the relation R is Deligne-Mumford. For any $x, y : X$ we know that:
  $$R(x, y) \simeq [x] =_{X/R} [y]$$
  so the direct direction is immediate. For the converse we use that being Deligne-Mumford is a proposition and that the map $[\_] : X \to X/R$ is surjective.
- The fibers of:
  $$[\_] : X \to X/R$$
  are formally étale if and only if the relation R is étale. For any $y : X$ we have that:
  $$\sum_{x:X} R(x, y) \simeq \mathrm{fib}_{[\_]}([y])$$
  so the direct direction is immediate. Here as well the converse follows from surjectivity of $[\_]$. ☐

Theorem 4.2.4
A type is an algebraic space if and only if it is merely the quotient of a scheme by a Deligne-Mumford étale equivalence relation.

Proof This is a direct application of proposition 4.2.3. ☐

4.3 Stability for algebraic spaces

Lemma 4.3.1 Algebraic spaces are stable by dependent sums.

Proof By proposition 2.2.4. ☐

Lemma 4.3.2 Algebraic spaces are stable by identity types.

Proof By definition. ☐

By lemma 4.3.1 and lemma 4.3.2, algebraic spaces are stable by finite limits.

Lemma 4.3.3 Algebraic spaces are stable by quotients by Deligne-Mumford étale equivalence relations.

Proof Assume given an algebraic space X. By proposition 4.2.3 it is enough to check that for any set Y which identity types are Deligne-Mumford with a formally étale surjection:

$$p : X \to Y$$

we have that Y is an algebraic space. Composing a scheme cover for X with p gives a scheme cover for Y. □

4.4 Examples

Example 4.4.1 The scheme $\mathbb{A}^1$ quotiented by the relation which identifies x and $-x$ when $x \neq 0$ is an algebraic space.

Proof We need to show that the equivalence relation E generated by $E(x, -x)$ when $x \neq 0$ is Deligne-Mumford and étale. This equivalence relation is:

$$E(x, y) = (x = y) + (x \neq 0 \land x = -y)$$

It is clearly a scheme. To check that it is étale, for any $y : R$ we compute:

$$\sum_{x:X} (x = y) + (x \neq 0 \land x = -y) \simeq 1 + (y \neq 0)$$

which is indeed étale. □

Example 4.4.2 The scheme:

$$\sum_{x,y:R} xy = 0$$

quotiented by the relation which identifies $(x, 0)$ and $(0, x)$ when $x \neq 0$ is an algebraic space.

Proof We need to show that the equivalence relation E generated by $E((x, 0), (0, x))$ when $x \neq 0$ is Deligne-Mumford and étale. This equivalence relation is:

$$E((x, y), (x', y')) = (x = x' \land y = y') + (x \neq 0 \land x = y' \land x' = 0)$$

as $x \neq 0$ implies $y = 0$ since $xy = 0$. It is clearly a scheme. To check that it is étale, for any $x', y' : R$ such that $x'y' = 0$ we compute:

$$\sum_{x,y:R} xy = 0 \land E((x, y), (x', y')) \simeq 1 + (y' \neq 0)$$

which is indeed étale. □

4.5 Algebraic spaces and group actions

Definition 4.5.1 An action of a group G on a type X is free if for all $x, y : X$ the type:

$$\sum_{g:G} gx = y$$

is a proposition.

If X is a set this is the same as asking that for all $x : X$ we have that $gx = x$ implies $g = 1$.

Lemma 4.5.2 Let G be an étale group scheme acting freely on an algebraic space X. Then:

$$x, y : X \mapsto \sum_{g:G} gx =_X y$$

is a schematic étale equivalence relation.

Proof The type:

$$\sum_{g:G} gx =_X y$$

is a scheme because it is a dependent sum of schemes. For any $y : Y$ we have:

$$\sum_{x:X}\sum_{g:G} gx = y \simeq G$$

which is assumed étale. □

Corollary 4.5.3 Algebraic spaces are stabe by quotient by free action of étale group schemes. In particular quotient of schemes by free action of étale group scheme are algebraic spaces.

Lemma 4.5.4 Let G be a finite group acting on an unramified scheme X, then X/G is an algebraic space.

Proof We are considering the quotient of X by the equivalence relation:

$$R(x, y) = \exists(g : G).gx =_X y$$

this is a schematic relation (even open) because G is finite and identity types in X are open propositions as X is unramified.

Now we need to show that for all $y : X$ the type:

$$\sum_{x:X} \exists(g : G).gx =_X y$$

is formally étale:

- It is formally unramified because it is a subtype of X, which is assumed formally unramified.
- It is formally smooth because we have a surjection:

$$G \simeq \sum_{x:X}\sum_{g:G} gx =_X y \to \sum_{x:X} \exists(g : G).gx =_X y$$

  and G is finite so it is smooth. □

5 Scheme quotient and algebraic spaces

The main goal of this section is to show that not every algebraic space is a scheme, even when its identity types are schemes. To do this we work with scheme quotients.

5.1 Quotient of an affine scheme by a finite group action

In all this section we assume G a finite group acting on $\mathrm{Spec}(A)$, such that the algebra of invariant A^G is finitely presented. Our goal is to prove that:

$$f : \mathrm{Spec}(A) \to \mathrm{Spec}(A^G)$$

is universal among G-invariant maps from $\mathrm{Spec}(A)$ to a scheme.

Remark 5.1.1 This hypothesis on A^G f.p. might not be easy (or possible) to remove, for example consider $R[X]$ quotiented by the $\mathbb{Z}/2\mathbb{Z}$-action sending X to $-X$. If $0 \neq 2$ then:

$$(R[X])^{\mathbb{Z}/2\mathbb{Z}} \cong R[X^2]$$

and if $0 = 2$ the action is trivial and:

$$(R[X])^{\mathbb{Z}/2\mathbb{Z}} \cong R[X]$$

and it seems delicate to choose a presentation without using $0 \neq 2$ or $0 = 2$.

Lemma 5.1.2 For all $U : \mathcal{O}(\mathrm{Spec}(A))$ that is G-invariant, there merely exists $V : \mathcal{O}(\mathrm{Spec}(A^G))$ such that $f^{-1}(V) = U$.

Proof We proceed in three steps:

- First we prove that for any $a : A$, writing:

$$\prod_{g:G}(X - ga) = X^n + b_{n-1}X^{n-1} + \cdots + b_0$$

  we have:

$$\vee_{g:G} D(ga) = D(b_{n-1}, \cdots, b_0)$$

  Indeed if for all $g : G$ we have that ga is nilpotent, then the b_j are nilpotent as well as they are symmetric polynomials in ga. Conversely if all the b_j are nilpotent then for all $g : G$ we have:

$$(ga)^n + b_{n-1}(ga)^{n-1} + \cdots + b_0 = 0$$

  so that $(ga)^n$ is a sum of nilpotent elements, so it is nilpotent.

- Since $b_{n-1}, \cdots, b_0 : A^G$ we have:

$$V = D(b_{n-1}, \cdots, b^0) : \mathcal{O}(\mathrm{Spec}(A^G))$$

  such that:

$$f^{-1}(V) = \vee_{g:G} D(ga)$$

- Then given $D(a_1, \cdots, a_m) : \mathcal{O}(\mathrm{Spec}(A))$ that is G-invariant we have:

$$D(a_1, \cdots, a_m) = \vee_{g:G} D(ga_1, \cdots, ga_m)$$
$$= \vee_i \vee_{g:G} D(ga_i)$$

  But by the previous point, for all i we get $V_i : \mathcal{O}(\mathrm{Spec}(A^G))$ such that:

$$f^{-1}(V_i) = \vee_{g:G} D(ga_i)$$

  Then:

$$f^{-1}(\cup_i V_i) = D(a_1, \cdots, a_m)$$ □

- We conclude by using the fact that any $U : \mathcal{O}(\mathrm{Spec}(A))$ is merely of the form $D(a_1, \cdots, a_m)$ for $a_1, \cdots, a_m : A$.

Lemma 5.1.3 Assume given $U, V : \mathcal{O}(\mathrm{Spec}(A^G))$ such that $f^{-1}(U) \subset f^{-1}(V)$. Then $U \subset V$.

Proof It is enough to prove the result when $U = D(a)$ and $V = D(b_1, \cdots, b_m)$ for $a, b_1, \cdots, b_m : A^G$. In this case the hypothesis $f^{-1}(U) \subset f^{-1}(V)$ means that there is m and $c_1, \cdots, c_m : A$ such that:

$$a^m = \sum_i c_i b_i$$

To prove $U \subset V$ we need the same thing for some $c_i : A^g$. But for all $g : G$ we have:

$$a^m = (ga)^m = \sum_i (gc_i) b_i$$

so that for n the cardinal of G we have:

$$a^{mn} = \prod_{g:G}\sum_i (gc_i) b_i$$

Consider $i_1, \cdots, i_n : \{1, \cdots, m\}$, it is enough to describe $d_{i_1,\cdots,i_n}$ the coefficient in front of $b_{i_1} \cdots b_{i_n}$ in the development of the right hand side, and to prove that it is G-invariant in order to conclude.

Consider Z the type of $\psi : G \to \{1, \cdots, m\}$ whose image is $i_1, \cdots, i_n$ counting multiplicities. Then:

$$d_{i_1,\cdots,i_n} = \sum_{\psi:Z}\prod_{g:G} gc_{\psi(g)}$$

To see it is invariant we remark that any $g : G$ acts on Z sending ψ to:

$$\psi_g : g' \mapsto \psi(g^{-1}g')$$

Then we have that:

$$\begin{aligned} g'd_{i_1,\cdots,i_n} &= \sum_{\psi:Z}\prod_{g:G} g'gc_{\psi(g)} \\ &= \sum_{\psi:Z}\prod_{h:G} hc_{\psi_{g'}(h)} \end{aligned}$$

by using the change of variable $h = g'g$. □

Lemma 5.1.4 For all $V : \mathrm{Spec}(A^G)$ the map:

$$f : f^{-1}(V) \to V$$

is an affine scheme quotient by the G-action.

Proof We proceed in two steps:

- We prove that for any algebra B and $b : B^G$, the canonical map:

  $$(B^G)_b \to (B_b)^G$$

  is an equivalence. It is clear that it is an embedding. We show that it is surjective. Assume:

  $$\frac{c}{b^n} : (B_b)^G$$

  Then for all $g : G$ we have that there exists m such that:

  $$b^m(c - gc) = 0$$

  Since g is finite we can take the sup of such m to get l such that for all $g : G$ we have:

  $$b^l(c - gc) = 0$$

  Then we have:

  $$\frac{c}{b^n} = \frac{b^l c}{b(l+n)} : (B_b)^G$$

  and $b^l c : B^G$ so we can conclude.

- Now any open V in $\mathrm{Spec}(A^G)$ is of the form $D(a_1, \cdots, a_n)$ with $a_1, \cdots, a_n : A^G$ and we can apply the previous point n-times to get a canonical equivalence:

  $$(A_{a_1,\cdots,a_n})^G \cong (A^G)_{a_1,\cdots,a_n}$$

  so it is enough to prove that:

  $$\mathrm{Spec}(B) \to \mathrm{Spec}(B^G)$$

  is an affine scheme-quotient for any f.p. B. This holds because of the duality between algebras and affine schemes. □

Lemma 5.1.5 Assume given Y a set with a dependent set $P(y)$ for $y : Y$. Assume given an open cover $(V_i)_{i:I}$ such that:

- For all $i : I$ we have that:

  $$\prod_{y:V_i} P(y)$$

  is contractible.

- For all $i, j : I$ we have that:

  $$\prod_{y:V_i\cap V_j} P(y)$$

  is contractible.

Then:

$$\prod_{y:Y} P(y)$$

is contractible.

Proof Omitted. Plain HoTT. □

Proposition 5.1.6 Assume G a finite group acting on an affine scheme $\mathrm{Spec}(A)$ such that A^G is a f.p. algebra. Then the map:

$$f : \mathrm{Spec}(A) \to \mathrm{Spec}(A^G)$$

is the scheme quotient of $\mathrm{Spec}(A)$ by the action of G.

Proof Assume given a G-invariant map from $\mathrm{Spec}(A)$ to a scheme X, we want to prove there is a unique dotted lift in:

$$\begin{array}{ccc} \mathrm{Spec}(A) & \xrightarrow{f} & \mathrm{Spec}(A^G) \\ & \searrow^{g} & \vdots \\ & & X \end{array}$$

We cover X by affine schemes U_i. Then using lemma 5.1.2 for all i we choose V_i such that:

$$f^{-1}(V_i) = g^{-1}(U_i)$$

By lemma 5.1.3 we know that the V_i cover $\mathrm{Spec}(A^G)$.

By lemma 5.1.5 it is enough to prove that there is a unique lifting over any V_i and over any $V_i \cap V_j$ in order to conclude.

- Let's prove this for V_i, assume given h a liftings:

  $$\begin{array}{ccc} f^{-1}(V_i) & \xrightarrow{f} & V_i \\ & \searrow^{g} & \downarrow h' \\ & & X \end{array}$$

  We check that:

  $$h(V_i) \subset U_i$$

  and the same with h'. This is equivalent to:

  $$V_i \subset h^{-1}(U_i)$$

  which by lemma 5.1.3 is equivalent to:

  $$f^{-1}(V_i) \subset f^{-1}h^{-1}(U_i)$$

  i.e.

  $$f^{-1}(V_i) \subset g^{-1}(U_i)$$

  which holds by definition. Then we have a triangle:

  $$\begin{array}{ccc} f^{-1}(V_i) & \xrightarrow{f} & V_i \\ & \searrow^{g} & \downarrow h \\ & & U_i \end{array}$$

  with U_i affine so that by lemma 5.1.4 there is a unique such h.

- Now for $V_i \cap V_j$, by the same reasoning we have:

  $$h(V_i \cap V_j) \subset U_i \cap U_j$$

  but $U_i \cap U_j$ is affine so we can conclude. □

5.2 Not all algebraic space are schemes

Let p be a prime number. We consider the action of:

$$\mu_p = \mathrm{Spec}(R[X]/X^p - 1)$$

on $\mathbb{A}^\times$ via multiplication.

Lemma 5.2.1 Assuming $p \neq 0$, the quotient of this action is an algebraic space.

Proof The polynomial $X^p - 1$ is separable when $p \neq 0$. So the scheme μ_p is étale. Moreover the action is free as it is multiplication by invertibles. So the quotient is an algebraic space. □

Lemma 5.2.2 Assume $a : R$ such that for all $j : \mu_p$ we have:

$$(1 - j)a = 0$$

Then $a = 0$.

Proof Through sqc, the assumption means that we have a map:

$$R[X]/X^p - 1, (1 - X)a \to R[X]/X^p - 1$$

sending X to X. So this means that $(1 - X)a = 0$ modulo $X^p - 1$, so that $a = 0$. □

Lemma 5.2.3 Assuming $p \neq 0$, and that μ_p is finite, the scheme-quotient of this action is the map:

$$\mathbb{A}^\times \to \mathbb{A}^\times$$

$$x \mapsto x^p$$

Proof We use proposition 5.1.6. We prove that the map:

$$R[X^p]_{X^p} \to (R[X]_X)^{\mu_p}$$

is an equivalence. It is clear that it is an embedding. We prove that it is surjective. Assume given:

$$\frac{P(X)}{X^n} : R[X]_X$$

such that for all $j : \mu_p$ we have:

$$\frac{P(X)}{X^n} = \frac{P(jX)}{(jX)^n}$$

Then writing:

$$P(X) = a_0 + \cdots + a_l X^l$$

we have for all k we have:

$$a_k j^{k-n} = a_k$$

By lemma 5.2.2 this implies that whenever $k \neq n$ modulo p, we have $a_k = 0$. So writing:

$$n = dp + r$$

our fraction is of the form:

$$\frac{a_r X^r + a_{r+p} X^{r+p} + \cdots + a_{ep+r} X^{ep+r}}{X^{dp+r}} = \frac{a_r + a_{r+p} X^p + \cdots + a_{ep+r} X^{ep}}{X^{dp}}$$

which is of the desired form. □

Lemma 5.2.4 Assuming $p \neq 0$, we not not have that μ_p is finite.

Proof Any separable polynomial can be factored into pairwise distinct linear components under a not-not. □

Proposition 5.2.5 Assuming $p \neq 0$, the quotient is an algebraic space but not a scheme.

Proof By lemma 5.2.1 the quotient is an algebraic space. Since we want to prove a negation, we can assume μ_p finite by lemma 5.2.4. If the quotient was a scheme, it would be equivalent to the scheme-quotient. Then by lemma 5.2.3 the map:

$$\mathbb{A}^\times \to \mathbb{A}^\times$$

$$x \mapsto x^p$$

would be surjective. This is a contradiction by lemma 3.5.1. □

Corollary 5.2.6 Some algebraic spaces are not schemes.

Proof The base ring is local, so either $2 \neq 0$ or $3 \neq 0$. In both cases we can conclude using proposition 5.2.5. □

6 Algebraic and Deligne-Mumford stacks

Definition 6.0.1 A 1-type X is a *algebraic stack*, if $x =_X y$ is an algebraic space for all $x, y : X$ and and there merely is a scheme U with a formally smooth surjection u onto X. If, in addition, there is a formally étale surjection $u : U \to X$, X is called a *Deligne-Mumford stack*.

Proposition 6.0.2 Let G be an étale group scheme acting on a scheme X, then the homotopy quotient $X//G$ is a Deligne-Mumford stack.

Proof The fibers of $X \to X//G$ are merely equivalent to G and therefore étale. This means the map $X \to X//G$ is formally étale. Let the action of G be given by a dependent type $\rho : BG \to \mathrm{Sch}_{\mathrm{qc}}$. The identity types in $X//G$ are of the form:

$$\begin{aligned} &(x, p) = (x', p') \\ \simeq &\sum_{g : x =_{BG} x'} \mathrm{transport}_\rho(g)(p) =_X p' \end{aligned}$$

So as a dependent sum of schemes over a scheme, the identity types are always scheme. □

7 Étale descent for algebraic stacks

This section is a draft toward a more proper definition of stacks, using étale sheaves. The main result so far is étale descent for algebraic stacks.

7.1 Étale sheaves and algebraic stacks

The following definition is local. It is supposed to be equivalent to the one using unramifiable polynomial, although we lack a proof at the moment.

Definition 7.1.1 A type X is an étale sheaf if is is $\|\mathrm{Spec}(A)\|$-local for all A fppf and étale algebra.

Remark 7.1.2 We conjecture this is equivalent to being local against monic unramifiable polynomials having roots. With the given definition we do not have a proof that all schemes are étale sheaf, although this certainly should hold.

Definition 7.1.3 A map $f : X \to Y$ is étale surjective if for all $y : Y$ we have:

$$\mathrm{et}(\|fib_f(y)\|)$$

Definition 7.1.4 An étale sheaf X has an étale atlas if there merely is an affine scheme $\mathrm{Spec}(A)$ and a map:

$$\mathrm{Spec}(A) \to X$$

that is formally étale and étale-surjective.

Definition 7.1.5 An étale sheaf X is an algebraic stack if:
- It has an étale atlas.

- Coinductively, its identity types are algebraic stacks.

An algebraic n-stack is a n-type that is an algebraic stack.

Remark 7.1.6 Note that an étale sheaf X being algebraic stack could be defined without coinduction using:

$$S_n(X) : \mathrm{Prop}$$

defined inductively on n by:

$$S_0(X) = \mathrm{hasEtaleAtlas}(X)$$
$$S_{n+1}(X) = \forall(x, y : X).\ S_n(x = y)$$

and asking:

$$\forall(n : \mathbb{N}).\ S_n(X)$$

7.2 Descent for algebraic stacks

Next lemma directly implies that Zariski cover are étale atlases, by taking $A = R$.

Lemma 7.2.1 If A is fppf and étale and we are given $f_1, \cdots, f_n : A$ such that $(f_1, \cdots, f_n) = 1$, we have that:

$$A_{f_1} \times \cdots \times A_{f_n}$$

is fppf and étale.

Proof TODO □

Proposition 7.2.2 Let X be an étale sheaf, then the property:

$$\mathrm{hasEtaleAtlas}(X)$$

is an étale sheaf.

Proof Assume A an fppf étale algebra such that:

$$\|\mathrm{Spec}(A)\| \to \mathrm{hasEtaleAtlas}(X)$$

We just need to prove that:

$$\mathrm{hasEtaleAtlas}(X)$$

Then we have:

$$\mathrm{Spec}(A) \to \mathrm{hasEtaleAtlas}(X)$$

and by Zariski local choice there merely is a Zariski cover $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ with for all $x : \mathrm{Spec}(A')$ an étale atlas:

$$f_x : \mathrm{Spec}(B_x) \to X$$

Then the induced map:

$$\sum_{x:\mathrm{Spec}(A')} \mathrm{Spec}(B_x) \to X$$

is an étale atlas for X, indeed its fiber over $z : X$ is:

$$\sum_{x:\mathrm{Spec}(A')} \sum_{y:\mathrm{Spec}(B_x)} f_x(y) = z$$

which is a sigma type of formally étale and étale inhabited types by lemma 7.2.1. □

Corollary 7.2.3 Let X be an étale sheaf, then the property:

$$\mathrm{isAlgStack}(X)$$

is an étale sheaf.

Proof Having X an étale sheaf asserts that all iterated identity types in X have étale atlases, but all these iterated identity types are étale sheaves so we can conclude using proposition 7.2.2 □

Next corollary asserts étale descent for algebraic stacks.

Corollary 7.2.4 The type of algebraic stack is an étale sheaf.

Proof The type of étale sheaf is an étale sheaf, and an étale sheaf X being an algebraic stack is an étale sheaf by corollary 7.2.3. □

7.3 Algebraic stacks are stable by quotients

Next lemma intuitively means that we can quotient algebraic stack by étale relation in the étale topos and get algebraic stack.

Lemma 7.3.1 Let X be a type such that:

- Identity types in X are algebraic stacks.
- X has an étale atlas.

Then $\text{et}(X)$ is an algebraic stack.

Proof We have X étale-sheaf separated by hypothesis and étale sheafification is lex so that:

$$i : X \subset \text{et}(X)$$

We denote the assumed atlas of X by:

$$f : \text{Spec}(B) \to X$$

- We have that $\text{et}(X)$ is an étale sheaf by definition.
- We have to prove that:

$$\prod_{x:\text{et}(X)} \text{isFormallyEtale}(\text{fib}_{i\circ f}(x))$$

  As affine schemes are étale sheaves, so is $\text{fib}_{i\circ f}(x)$, and so it being formally étale is an étale sheaf. Therefore it is enough to prove:

$$\prod_{x:X} \text{isFormallyEtale}(\text{fib}_{i\circ f}(i(x)))$$

  But since i is an embedding, we have that:

$$\text{fib}_{i\circ f}(i(x)) = \text{fib}_f(x)$$

  and the fiber of f is assumed formally étale.
- We have to prove that:

$$\prod_{x,y:\text{et}(X)} \text{isAlgStack}(x =_{\text{et}(X)} y)$$

  but by proposition 7.2.2, it is enough to prove that:

$$\prod_{x,y:X} \text{isAlgStack}(i(x) =_{\text{et}(X)} i(y))$$

  But:

$$(i(x) =_{\text{et}(X)} i(y)) \simeq \text{et}(x =_X y) \simeq (x =_X y)$$

  and $x =_X y$ is assumed to be a algebraic stack. □

Next corollary assume that schemes are étale sheaves, which we have not proven yet.

Corollary 7.3.2 Let X be a scheme and let $\sim$ be an equivalence relation on X such that:

- For all $x, y : X$, we have that $x \sim y$ is a scheme.
- For all $x : X$, the type:

$$\sum_{y:X} x \sim y$$

  is formally étale.

Then $\text{et}(X/\sim)$ is an algebraic set.

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