Sheaves in Synthetic Algebraic Geometry

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The following is a very rough draft containing some preliminary works on sheaves in the context of synthetic algebraic geometry. It is written by Hugo Moeneclaey. Section 1 contains results very similar to Section 10 of foundations.

Contents

1 Sheaves in the presheaf topos

In this section, we work in the topos of presheaf on the Zariski site, meaning that I use a (non-local) ring R with synthetic coherence:

$$A \simeq R^{\mathrm{Spec}(A)}$$

for A f.p. R-algebra, as well as global choice for affine schemes. It is very similar to Section 10 of fundamentals, although a bit more general in its phrasing.

1.1 Preliminaries

Lemma 1.1.1 Affine schemes are closed by dependent sums.

Proof Assume given B_x a f.p. R algebra depending on $x : \mathrm{Spec}(A)$. Using boundedness and choice, as well as the stability of affine schemes by binary sums, we may assume that we merely have:

$$B_x = R[X_1, \cdots, X_n]/(P_1(x), \cdots, P_m(x))$$

where the coefficients of $P_1, \cdots, P_m$ depends on $x : \mathrm{Spec}(A)$. By synthetic quasi-coherence this gives an f.p. R algebra:

$$B = A[X_1, \cdots, X_n]/(P_1, \cdots, P_m)$$

Then we have that:

$$\mathrm{Spec}(B) = \sum_{x:\mathrm{Spec}(A)} \mathrm{Spec}(B_x)$$

as the canonical map:

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

has fiber $\mathrm{Spec}(B_x)$ over $x : \mathrm{Spec}(A)$. □

Lemma 1.1.2 Finite sums of affine schemes are closed by dependent sums.

Proof It is enough to prove this for an affine scheme $\mathrm{Spec}(A)$ and a family B_x of finite sum of affine schemes indexed by $x : \mathrm{Spec}(A)$.

Using boundedness and choice for $\mathrm{Spec}(A)$ we merely get a map:

$$f : \mathrm{Spec}(A) \to \{0, \cdots, n\}$$

sending x to the number of summed affine scheme in B_x. By considering:

$$\mathrm{Spec}(A) = f^{-1}(0) + \cdots + f^{-1}(n)$$

we see that it is enough to prove the result when f is constant, and to treat this case it is enough to prove the result for a family of affine schemes over $\mathrm{Spec}(A)$. This is lemma 1.1.1. □

1.2 Definitions of sheaves

Definition 1.2.1 A pre-topology is a class T of finite sums of affine schemes. It is called a topology if T is closed under dependent sums and $1 \in T$.

Remark 1.2.2 We need to use finite sums of affine schemes rather than just affine schemes to accommodate the fact that $\perp$ is not an affine scheme when $0 = 1$ in R. Note that such a finite sum has choice, as it is either empty or an affine scheme.

Intuitively, an affine scheme is in T if it covers the point. There is an obvious topology generated by any pre-topology

Remark 1.2.3 For the notion of topology to make sense we need finite sums of affine scheme to be closed under dependent sums, as shown in lemma 1.1.2.

Definition 1.2.4 Let T be a pretopology. A map is a T-cover if its fibers are in T.

If T is a topology, T-covers are stable by composition.

Definition 1.2.5 Let T be a pretopology. A type is a T-sheaf if it is $\|X\|$-local for all $X \in T$. We write L_T for the T-sheafification.

Lemma 1.2.6 Being a T-sheaf is equivalent to being a sheaf for the topology engendered by T.

Proof Let us denote by T' the class of X such that $L_T\|X\|$ holds. Then any T-sheaf Z is $\|X\|$-local for any $X \in T'$ as then:

$$Z^{\|X\|} \simeq Z^{L_T\|X\|} \simeq Z$$

We want to show that the topology engendred by T is included in T'. It is clear that $T \subset T'$, so we just need to prove that T' is stable by dependent sum. This holds because T' is the type of connected types for a modality. □

Remark 1.2.7 Being a T-sheaf is a lex modality, so that T-sheaves are stable by dependent sum, identity types and product over arbitrary types, and the type of T-sheaves is a T-sheaf.

This means that HoTT can be interpreted into T-sheaf, by replacing the universe of types by the universe of T-sheaves. This interpretation can be extended to propositional truncation by sending the propositional truncation of a T-sheaf X to:

$$L_T\|X\|$$

1.3 T-sheaves agree with the usual sheaves

Lemma 1.3.1 For X a set and any type A, we have that the canonical map:

$$X^{\|A\|} \to \sum_{f:X^A} \prod_{a,b:A} f(a) = f(b)$$

is an equivalence.

Proof To define the map the other way, assume given such an $f : A \to X$. We consider the family of propositions:

$$\_ : \|A\| \mapsto \sum_{x:X} \prod_{a:A} f(a) =_X x$$

To show that it has a section it is enough to prove it has a section for any $a : A$, which is clear using $f(a) : X$. We omit the proof that these maps are inverse to each other. □

Next lemma prove that a type X is a T-sheaf iff it is a sheaf in the usual sense, at least when X is a set.

Lemma 1.3.2 Let T be a pre-topology, then a set X is a T-sheaf if and only if for any T-cover:

$$A \to B$$

we have that the canonical map:

$$X^B \to \lim(X^A \rightrightarrows X^{A\times_B A})$$

is an equivalence.

Proof Since dependent sums commute with limits, we can reason fiberwise over B and then the given condition is equivalent to the canonical maps:

$$X \to \lim(X^A \rightrightarrows X^{A\times A})$$

being an equivalence for all A in T.

To conclude that this condition is equivalent to being a T-sheaf we just need to prove that the canonical maps:

$$X^{\|A\|} \to \lim(X^A \rightrightarrows X^{A\times A})$$

are equivalences, but this is lemma 1.3.1. □

This could in principle be extended to n-types for any n, given enough patience to write down all the coherences. There is probably a way using the approximation for propositional truncation using iterated join.

1.4 Sheaf models compared to the presheaf model

Next lemma says that the interpretation of $\|X\|$ into T-sheaves holds for all $X \in T$.

Lemma 1.4.1 Let T be a pre-topology, then for any $X \in T$, we have that:

$$L_T\|X\|$$

Proof We have that:

$$\|X\| \to L_T\|X\|$$

and that $L_T\|X\|$ is $\|X\|$-local, so $L_T\|X\|$ holds. □

Definition 1.4.2 A pre-topology T is called subcanonical if R is a T-sheaf.

Lemma 1.4.3 If T is a subcanonical pre-topology, then the intepretation of synthetic quasi-coherence holds in T-sheaves.

Proof Since R is a T-sheaf, so is any affine scheme, and by synthetic quasi-coherence so is any f.p. R-algebra. Then the interpretation of synthetic quasi-coherence into T-sheaves is just the usual synthetic quasi coherence, which is assumed to hold. □

Lemma 1.4.4 If T is a topology and P is a proposition, then

$$L_T(P) \simeq \exists X \in T.\ P^X$$

Proof First we check that if:

$$\exists X \in T.\ P^X$$

we have $L_T(P)$. Since the goal is a proposition (as modality preserves propositions), we can assume $X \in T$ such that $X \to P$. Then we have that:

$$\|X\| \to P \to L_T(P)$$

so that $L_T(P)$ holds as it is $\|X\|$-local.

Conversely we have that:

$$P \to \exists X \in T.\ P^X$$

as $1 \in T$ so it is enough to check that the proposition $\exists X \in T.\ P^X$ is T-local. Assume $Y \in T$ such that:

$$\|Y\| \to \exists X \in T.\ P^X$$

then we have:

$$Y \to \exists X \in T.\ P^X$$

and since Y has choice we merely have:

$$Y \to \sum_{X \in T} P^X$$

so that we merely have $X_y \in T$ depending on $y : Y$ such that:

$$(\sum_{y:Y} X_y) \to P$$

Since T is closed by dependent sum we can conclude that:

$$\exists X \in T.\ P^X$$

□

Definition 1.4.5 We say that a type X has T-local choice if for all $P(x)$ depending on $x : X$ with:

$$\prod_{x:X} \|P(x)\|$$

there merely exists a T-cover $f : Y \to X$ such that:

$$\prod_{y:Y} P(f(y))$$

Next lemma implies that when T is a topology and X has choice, the interpretation of X having T-local choice holds in T-sheaves.

Lemma 1.4.6 If T is topology, for any type X enjoying choice and $P(x)$ depending on $x : X$, if we have:

$$\prod_{x:X} L_T\|P(x)\|$$

then there merely exists a T-cover $f : Y \to X$ such that:

$$\prod_{y:Y} P(f(y))$$

Proof By lemma 1.4.4 the hypothesis is equivalent to:

$$\prod_{x:X} \exists X \in T. P(x)^X$$

then X has choice so we can conclude. □

We bundle all of this in one result:

Theorem 1.4.7
If T is a subcanonical toplogy, then the interpretation of the following holds in T-sheaves:

(i) We have $\|X\|$ for all $X \in T$.

(ii) Synthetic quasi-coherence.

(iii) Any affine scheme has T-local choice.

Proof (i) is by lemma 1.4.1.
(ii) is by lemma 1.4.3.
(iii) is by lemma 1.4.6. □

1.5 Zariski topology

Definition 1.5.1 An affine scheme is in the Zariski pre-topology Zar if it merely is of the following form:

$$\mathrm{Spec}(R_{f_1}) + \cdots + \mathrm{Spec}(R_{f_n})$$

for $f_1, \cdots, f_n : R$ with $(f_1, \cdots, f_n) = 1$.

Lemma 1.5.2 The Zariski pre-topology is a toplogy.

Proof TODO, should use the fact that any map $\mathrm{Spec}(R_f) \to \mathbb{N}$ gives a partition of R_f by a fundamental system of idempotents. □

Lemma 1.5.3 The Zariski topology is subcanonical.

Proof Using lemma 1.3.2 we just need to prove that given $f_1, \cdots, f_n$ generating R, giving $x : R$ is equivalent to giving a family $x_i : R_{f_i}$ of pairwise compatible elements.
TODO □

Now we can state the main theorem about Zariski-sheaves, namely that they model synthetic algebraic geometry.

Theorem 1.5.4
The interpretation of the following in Zariski-sheaves holds:

(i) The ring R is local.

(ii) Synthetic quasi-coherence.

(iii) Any affine scheme has Zariski-local choice.

Proof By theorem 1.4.7 it is enough to prove that:

$$\|X\|$$

for any X in the Zariski topology implies that R is local.
If $0 = 1$ then the empty sum is in Zariski, which gives a contradiction. For any $f : R$ we have that:

$$\mathrm{Spec}(R_f) + \mathrm{Spec}(R_{1-f})$$

is in the Zariski topology so that:

$$\mathrm{Spec}(R_f) \vee \mathrm{Spec}(R_{1-f})$$

holds, meaning that R is local. □

1.6 Toward étale topology

We give a model where every separable monic polynomial has a root and affine schemes have étale-local choice. We call types in this modal separable sheaves.

Remark 1.6.1 In the étale topos we expect to get the a priori stronger axiom that every unramifiable polynomial has a root. We do not know the relationship between separable and étale sheaves.

Definition 1.6.2 The separable topology is the topology generated by the Zariski topology together with:

$$\mathrm{Spec}(R[X]/g)$$

for any g monic separable (i.e. g' invertible in $R[X]/g$).

Lemma 1.6.3 Any separable cover is formally étale.

Proof Since formally étale types are closed under dependent sum and 1 is formally étale, it is enough to prove that finite sums of open proposition are formally étale, and that $\mathrm{Spec}(R[X]/g)$ is formally étale for g monic separable. □

Lemma 1.6.4 Let $g : R[X]$ be a monic polynomial. Then R is $\|\mathrm{Spec}(R[X]/g)\|$-local.

Proof Using lemma 1.3.2 we just need to prove that the map from R to the equaliser of:

$$R[X]/g \rightrightarrows R[X]/g \otimes R[X]/g$$

is an equivalence. But since g is monic we merely have:

$$R[X]/g \simeq R^n$$

and then we check that the induced map from R to the equaliser of:

$$R^n \rightrightarrows R^n \otimes R^n$$

is an equivalence. □

Lemma 1.6.5 The separable topology is subcanonical.

Proof By lemma 1.2.6 it is enough to show that it is a sheaf for the pretopology generating the separable topology. By lemma 1.5.3 we know that R is a Zariski-sheaf. By lemma 1.6.4 we can conclude. □

Theorem 1.6.6
The interpretation of the following in separable-sheaves holds:

(i) The ring R is local. Any monic separable polynomial in $R[X]$ merely has a root.

(ii) Synthetic quasi-coherence.

(iii) Any affine scheme $\mathrm{Spec}(A)$ has étale-local choice (meaning that for any family P of inhabited types we have an étale surjective map to $\mathrm{Spec}(A)$ over which P has a section).

Proof We apply theorem 1.4.7. We still need to prove is that a separable cover gives a surjective formally étale map inside separable sheaves. It is clear that it is surjective, and it is formally étale by lemma 1.6.3 (as being formally étale means the same in the presheaf topos or in T-sheaves). □

2 Sheaves in the Zariski topos

In this section we work in with Zariski sheaves, i.e. the usual synthetic algebraic geometry. A lot of it is direct reformulation of the previous section.

2.1 Definition of sheaves

Definition 2.1.1 A pre-topology T is a class of affine schemes.

Definition 2.1.2 A topology T is a class of affine schemes such that:

- It is stable by dependent sums and contains the unit.
- It is stable by Zariski cover, meaning that given a Zariski cover:

  $$\sum_{i:I} U_i \to X$$

  if $X \in T$ then $\sum_{i:I} U_i \in T$.

There is a smallest topology generated by a pre-topology.

Definition 2.1.3 Let T be a pre-topology, then a type is a T-sheaf if it is $\|X\|$-local for all $X \in T$.

This agree with the usual definition of sheaves at least for sets (and presumably for all n-types for a finite n). We write L_T for T-sheafification, which is a lex modality.

Lemma 2.1.4 Let T be a topology and let T' be the smallest topology containing T. A type is a T-sheaf if and only if it is a T'-sheaf.

Proof We proceed as in lemma 1.2.6. The only new result we need is that is that the class of affine schemes X such that $L_T\|X\|$ is closed by Zariski cover. But if:

$$Y \to X$$

is a Zariski cover then:

$$\|Y\| \leftrightarrow \|X\|$$

so we can conclude. □

2.2 Covers and choice

Lemma 2.2.1 Let T be a topology and P be a proposition, then:

$$L_T(P) \leftrightarrow \exists X \in T.\ P^X$$

Proof We proceed as in lemma 1.4.4, using Zariski-local choice and the stability of T by Zariski cover instead of choice. □

Definition 2.2.2 Let T be a pre-topology. A T-cover of an affine scheme is inductively defined by:

- The identity is a T-cover, and the composite of T-covers is a T-cover.
- A map:

  $$f : X \to Y$$

  is a T-cover if we have a Zariski cover $(U_i)_{i:I}$ of Y where the maps:

  $$f : f^{-1}(U_i) \to U_i$$

  are T-cover.
- A map which fibers are in T is a T-cover.

Lemma 2.2.3 For any pre-topology T, Zariski covers are T-cover.

Proof Zariski covers are Zariski-locally identity maps. □

Lemma 2.2.4 Let T be a pre-topology generating the topology T'. Then a map is a T-cover if and only if its fibers are in T'.

Proof First we prove inductively that T-covers have fibers in T'.

- If the T-cover is the identity, we can conclude as $1 \in T'$. If it is a composite we conclude using the stability of T' under dependent sums.

- Assume a map:

$$f : X \to Y$$

with a Zariski cover $(U_i)_{i:I}$ of Y where the induced maps:

$$f_i : f^{-1}(U_i) \to U_i$$

have fibers in T'. Then any $y : Y$ is in U_i for some i so the fiber of f over y is equivalent to the fiber of f_i, which is in T'.

- If the fibers of the T-cover are in T, then we can conclude as $T \subset T'$.

Now let us prove the converse. Assume given a map between affine schemes with fibers in T'. Zariski-locally we can assume we are in one of the following case:

- The fibers are 1. We can conclude because identity maps are T-covers.
- The fibers are dependent sum of things in T'. Here we conclude inductively from the fact that T-covers are stable under composition.
- The fibers are Zariski coverings of things in T'. Then the map is a composite of a Zariski cover followed by a T-cover. We conclude by lemma 2.2.3 and the stability of T-cover by composition.
- The fibers are in T. Then we conclude by definition. □

Lemma 2.2.5 Let T be a pre-topology, assume given an affine scheme $\mathrm{Spec}(A)$ and a family of types $P(x)$ for $x : A$. Assume:

$$\prod_{x:\mathrm{Spec}(A)} L_T \| P(x) \|$$

then there exists a T-cover:

$$f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

such that:

$$\prod_{y:\mathrm{Spec}(B)} P(f(y))$$

Proof Same as lemma 1.4.6 considering T' the topology generated by T, with lemma 2.2.4 and Zariski-local choice rather than choice. □

2.3 Sheaf models TODO

This section is incomplete, we would need to study the sheaf interpretation more in details (TODO).

Lemma 2.3.1 Let T be a pre-topology, then we have $L_T(\|X\|)$ for all $X \in T$.

Proof Immediate by lemma 2.2.1. □

Theorem 2.3.2
Let T be a subcanonical pre-topology, then T-sheaves enjoys the following:

(i) For any $X \in T$ we have $\|X\|$.

(ii) Synthetic quasi-coherence.

(iii) Affine schemes have T-local choice (meaning choice using T-covers).

Proof TODO □

3 Fppf sheaves using only monic polynomial

This section is concerned with fppf sheaves in the Zariski topos. It is outdated as it only uses roots of monic monic polynomial rather than every faithfully flat algebra.

3.1 Definition

Definition 3.1.1 The fppf topology is the topology generated by $\mathrm{Spec}(R[X]/g)$ for all monic $g : R[X]$.

Example 3.1.2 Finite types are fppf sheaves.

3.2 Schemes are fppf sheaves

Lemma 3.2.1 The type R is an fppf sheaf. In other words, the fppf topology is subcanonical.

Proof Using lemma 1.3.2 we just need to prove that the map from R to the equaliser of:

$$R[X]/g \rightrightarrows R[X]/g \otimes R[X]/g$$

is an equivalence. But since g is monic we merely have:

$$R[X]/g \simeq R^n$$

and then we check that the induced map from R to the equaliser of:

$$R^n \rightrightarrows R^n \otimes R^n$$

is an equivalence. □

Corollary 3.2.2 Any affine scheme is a fppf sheaf. Any f.p. R-algebra is an fppf sheaf.

Lemma 3.2.3 Assume given a type P such that:

- The type R is $\|P\|$-local.
- Any open proposition is $\|P\|$-local.
- The type of open propositions is $\|P\|$-smooth.

Then any scheme is $\|P\|$-local.

Proof Since R is $\|P\|$-local we see that all affine schemes are $\|P\|$-local through stability under dependent products and identity types.

We check that any scheme X is $\|P\|$-smooth. Assume given constant map:

$$f : P \to X$$

Take $(U_i)_{i:I}$ a finite cover of X by affine scheme. Then for any $i : I$ we have that $f^{-1}(U_i)$ is a constant open in P, so since the type of open is $\|P\|$-smooth, we merely have an open proposition V_i such that for all $x : P$, we have:

$$(x \in f^{-1}(U_i)) \leftrightarrow V_i$$

Since the $f^{-1}(U_i)$ cover P, we have that:

$$P \to \vee_{i:I} V_i$$

But open propositions are assumed are $\|P\|$-local, we have that:

$$\vee_{i:I} V_i$$

Assume $k : I$ such that V_k holds. Then $f^{-1}(U_k) = P$ and the constant map f factors through the affine scheme U_k. Since affine schemes are $\|P\|$-local, we merely have a lift for f.

Now we conclude that any scheme is $\|P\|$-local by proving that its identity types are $\|P\|$-local. Indeed they are propositional schemes, so they are $\|P\|$-smooth propositions, so they are $\|P\|$-local. □

Lemma 3.2.4 For any monic $g : R[X]$, we have that $\mathrm{Spec}(R[X]/g)$ is projective. In particular it is compact, meaning that for any open U in $\mathrm{Spec}(R[X]/g)$ the proposition:

$$\prod_{x:\mathrm{Spec}(R[X]/g)} U(x)$$

is open.

Proof Assume that:

$$g = X^n + a_{n-1}X^{n-1} + \cdots + a_0$$

Then we consider the homogeneous polynomial:

$$f(X,Y) = X^n + a_{n-1}X^{n-1}Y + \cdots + a_0Y^n$$

We prove that:

$$\sum_{[x,y]:\mathbb{P}^1} f(x,y) = 0$$

is equivalent to $\mathrm{Spec}(R[X]/g)$. Indeed for any $x, y : R$ such that $f(x,y) = 0$, we have that $x \neq 0$ implies $y \neq 0$, so that $(x,y) \neq 0$ implies $y \neq 0$. Then:

$$\sum_{[x,y]:\mathbb{P}^1} f(x,y) = 0$$

is equivalent to:

$$\sum_{x:R} f(x,1) = 0$$

which is the type of roots of g.

Now we conclude using the fact that

$$\sum_{[x,y]:\mathbb{P}^1} f(x,y) = 0$$

is closed in the compact scheme $\mathbb{P}^1$, so that it is compact. □

Remark 3.2.5 There should be an alternative computational proof along the line of

$$\prod_{x:\mathrm{Spec}(R[X]/g)} f_1(x) \neq 0 \vee \cdots \vee f_n(x) \neq 0$$

is equivalent to an explicit open proposition depending on the f_i and the coefficient of g.

Proposition 3.2.6 Any scheme is an fppf sheaf.

Proof Assume given $g : R[X]$ monic, by lemma 3.2.3 it is enough to prove that R is $\|\mathrm{Spec}(R[X]/g)\|$-local (this is lemma 3.2.1), that open propositions are $\|\mathrm{Spec}(R[X]/g)\|$-local and that the type of open propositions is $\|\mathrm{Spec}(R[X]/g)\|$-smooth.

To prove that open propositions are $\|\mathrm{Spec}(R[X]/g)\|$-local, we assume a map:

$$\|\mathrm{Spec}(R[X]/g)\| \to U$$

but we have $\neg\neg\,\mathrm{Spec}(R[X]/g)$ and $\neg\neg U \to U$ so we can conclude.

Assume given a constant open $D(h_1, \cdots, h_n)$ in $\mathrm{Spec}(R[X]/g)$. Then for any $x : \mathrm{Spec}(R[X]/g)$ we have that:

$$x \in D(h_1, \cdots, h_n) \leftrightarrow \prod_{y:\mathrm{Spec}(R[X]/g)} y \in D(h_1, \cdots, h_n)$$

because the open $D(h_1, \cdots, h_n)$ is constant. But the right hand side is open by lemma 3.2.4, so we indeed have a lift. □

3.3 Fppf covers are flat

Definition 3.3.1 A map between affine schemes:

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is flat if for all injective morphism of A-modules:

$$M \to N$$

the induced map:

$$B \otimes_A M \to B \otimes_A N$$

is injective.

Lemma 3.3.2 Assume a f.p. R-algebra A and $g : A[X]$ monic, then the induced map:

$$\mathrm{Spec}(A[X]/g) \to \mathrm{Spec}(A)$$

is flat.

Proof Since g is monic we have:

$$A[X]/g \simeq A^n$$

as A-modules. Then given an injective map of A-modules:

$$f : M \to N$$

the induced map;

$$A[X]/g \otimes_A M \to A[X]/g \otimes_A N$$

is of the form:

$$f^n : M^n \to N^n$$

which is injective. □

Lemma 3.3.3 For any f.p. R-algebra A and any $f : A$, the map:

$$\mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$$

is flat.

Proof Assume given an injective map of A-modules:

$$g : M \to N$$

Assume given:

$$\frac{m}{f^i}, \frac{n}{f^j}$$

in $A_f \otimes_A M$ such that:

$$g(\frac{m}{f^i}) = g(\frac{n}{f^j})$$

in $A_f \otimes_A N$. Then there is k such that:

$$f^{j+k} g(m) = f^{i+k} g(n)$$

in N. But then since g is A-linear we have:

$$g(f^{j+k} m) = g(f^{i+k} n)$$

in N, so from injectivity of g we conclude:

$$f^{j+k} m = f^{i+k} n$$

which implies that:

$$\frac{m}{f^i} = \frac{n}{f^j}$$

in $A_f \otimes_A M$. □

We have a kind of converse:

Lemma 3.3.4 Let A be a f.p. R algebra and $f_1, \cdots, f_n : A$ such that $(f_1, \cdots, f_n) = A$. Assume given a map between between A-modules:

$$g : M \to N$$

such that for all i the induced map:

$$A_{f_i} \otimes_A M \to A_{f_i} \otimes_A N$$

is injective. Then g is injective.

Proof Assume m in M such that $g(m) = 0$ in N. Then $g(m) = 0$ in $A_{f_i} \otimes_A N$ so by the assumed injectivity we have $m = 0$ in $A_{f_i} \otimes_A M$ for all i.

This means that for all i we have $k_i : \mathbb{N}$ such that $f_i^{k_i} m = 0$ in M. But then since the $D(f_i^{k_i})$ cover $\mathrm{Spec}(A)$ we know that $(f_1^{k_1}, \cdots, f_n^{k_n}) = A$ and we can conclude that $m = 0$ in M as we needed. □

Lemma 3.3.5 Being flat is Zariski-local in the target. More precisely assume given a map between affine schemes:

$$f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

and a Zariski cover $(U_i)_{i:I}$ of $\mathrm{Spec}(A)$. If for all $i : I$ the map:

$$f : f^{-1}(U_i) \to U_i$$

is flat, then the map f is flat.

Proof Assume an injective map betweem f.p. A-module:

$$M \to N$$

For all i we have an injective map between f.p. A_{f_i}-module:

$$A_{f_i} \otimes_A M \to A_{f_i} \otimes_A N$$

because localisation is flat by lemma 3.3.3. Since:

$$\mathrm{Spec}(B \otimes_A A_{f_i}) \to \mathrm{Spec}(A_{f_i})$$

is flat, we know that:

$$A_{f_i} \otimes_A B \otimes_A M \to A_{f_i} \otimes_A B \otimes_A N$$

is injective for all i, and from lemma 3.3.4 we conclude that the map:

$$B \otimes_A M \to B \otimes_A N$$

is injective. □

Lemma 3.3.6 Any fppf cover is flat.

Proof We proceed by induction on the fppf cover:

- If the fppf cover is an identity or a composite of fppf cover, we just need to check that identity maps are flat and that flat maps are stable under composition.
- If the fppf cover is Zariski-locally an fppf cover we use lemma 3.3.5.
- If the fppf cover has fibers of the form $\mathrm{Spec}(R[x]/g)$ with g monic, then we know that it is Zariski-locally of the form:

  $$\mathrm{Spec}(B[X]/f) \to \mathrm{Spec}(B)$$

  with f monic and we conclude by lemma 3.3.5 and lemma 3.3.2. □

3.4 Fppf covers are faithfully flat

Definition 3.4.1 A map between affine schemes:

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is faithfully flat if it is flat and for all A-module M we have that $B \otimes_A M = 0$ implies $M = 0$.

Lemma 3.4.2 Assume a f.p. R-algebra A and $g : A[X]$ monic, then the induced map:

$$\mathrm{Spec}(A[X]/g) \to \mathrm{Spec}(A)$$

is faithfully flat.

Proof It is flat by lemma 3.3.2. Assume M an A-module such that:

$$A[X]/g \otimes_A M = 0$$

then since $A[X]/g = A^n$ as A-modules we have $M^n = 0$ so that $M = 0$. □

Lemma 3.4.3 Assume given a pullback square of affine schemes:

$$\begin{array}{ccc} \mathrm{Spec}(B \otimes_A C) & \longrightarrow & \mathrm{Spec}(B) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(C) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

If f is flat so is g. If f is faithfully flat so is g.

Proof Assume f flat. Then for all injective map of C-modules:

$$M \to N$$

we have that:

$$B \otimes_A M \to B \otimes_A N$$

so that:

$$M \to N$$

is injective as a map of A-modules, so it is injective as a map of C-modules as well.

Now assume f faithfully flat. If M is a C-module such that:

$$B \otimes_A C \otimes M = 0$$

i.e. $B \otimes_A M = 0$, then $M = 0$ as an A-module, so $M = 0$ as a C-module. □

Lemma 3.4.4 Being faithfully flat is Zariski-local in the target. More precisely assume given a map between affine schemes:

$$f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

and a Zariski cover $(U_i)_{i:I}$ of $\mathrm{Spec}(A)$. If for all $i : I$ the map:

$$f : f^{-1}(U_i) \to U_i$$

is faithfully flat, then the map f is faithfully flat.

Proof The map is flat by lemma 3.3.5. Now assume an A-module M such that $B \otimes_A M = 0$. Up to refinement, using lemma 3.4.3, we can assume that $U_i = D(f_i)$ for all i, with the f_i generating A. Then we have:

$$B \otimes_A A_{f_i} \otimes_{A_{f_i}} A_{f_i} \otimes M = 0$$

so that since by hypothesis:

$$\mathrm{Spec}(B \otimes_A A_{f_i}) \to \mathrm{Spec}(A_{f_i})$$

is faithfully flat, for all i we have that:

$$A_{f_i} \otimes_A M = 0$$

Then we conclude using lemma 3.3.4 on the map $M \to 0$, concluding that it is injective so that $M = 0$. □

Proposition 3.4.5 Any fppf cover is faithfully flat.

Proof It is flat by lemma 3.3.6. The rest the proof for faithfulness is very similar. More precisely, to see that it is faithfully flat we proceed by induction on the fppf cover:

- If the fppf cover is an identity or a composite of fppf cover, we just need to check that identity maps are faithfully flat and that faithfully flat maps are stable under composition.
- If the fppf cover is Zariski-locally an fppf cover we use lemma 3.4.4.

- If the fppf cover has fibers of the form $\mathrm{Spec}(R[x]/g)$ with g monic, then we know that it is Zariski-locally of the form:

$$\mathrm{Spec}(B[X]/f) \to \mathrm{Spec}(B)$$

  with f monic and we conclude by lemma 3.4.4 and lemma 3.4.2. □

Lemma 3.4.6 Assume an affine scheme $\mathrm{Spec}(A)$ with a family of types $P(x)$ for $x : \mathrm{Spec}(A)$ such that:

$$\prod_{x:\mathrm{Spec}(A)} L_{fppf}\|P(x)\|$$

Then there merely exists a faithfully flat map:

$$f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

such that:

$$\prod_{x:\mathrm{Spec}(B)} P(f(x))$$

Proof By lemma 2.2.5 there exists an such an f that is an fppf cover. We conclude by proposition 3.4.5.□

Proposition 3.4.7 Any flat and fppf surjective map between affine schemes is faithfully flat.

Proof We have an fppf surjective map:

$$f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

meaning that:

$$\prod_{x:\mathrm{Spec}(A)} L_{fppf}\|\mathrm{fib}_f(x)\|$$

Then by lemma 3.4.6 we have a commutative diagram:

$$\begin{array}{ccc} & \nearrow & \mathrm{Spec}(B) \\ & & \downarrow f \\ \mathrm{Spec}(C) & \xrightarrow[g]{} & \mathrm{Spec}(A)git \end{array}$$

where g is faithfully flat. It is easy to conclude from this. □

3.5 The fppf sheaf model TODO

Some work need to be done on interpretation of statement into modal types (for a lex modality).

Theorem 3.5.1
The following holds when interpreted in fppf sheaves:

(i) The ring R is local, and any monic polynomial in R merely has a root.

(ii) Synthetic quasi coherence.

(iii) Affine schemes enjoys flat local choice, meaning that given TODO

Proof TODO □

4 Fppf sheaves revisited

This section is about fppf sheaves.

4.1 Flat and faithfully flat modules

Definition 4.1.1 An A-module M is flat if $\_ \otimes_A M$ preserve monomorphisms of A-module.

Lemma 4.1.2

4.2 Flat and faithfully flat algebras

Definition 4.2.1 Given a map between f.p. algebras $A \to B$ we say that B is a flat (resp. faithfully flat) A-algebra if B is flat (resp. faithfully flat) as an A-module.

Then we say that the induced map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat (resp. faithfully flat).

The following two propositions are based on Lombardi:

Proposition 4.2.2 Assume given a map of fp algebras $\rho : A \to B$, then the following are equivalent:

(i) B is a flat A algebra.

(ii) TODO

Proof TODO □

Proposition 4.2.3 Assume given a map of fp algebras $\rho : A \to B$, then the following are equivalent:

(i) B is a faithfully flat A algebra.

(ii) TODO

Proof TODO □

Lemma 4.2.4 A map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat if (resp. faithfully flat) and only if its fibers are flat (resp. faithfully flat) R-algebras.

Proof TODO □

Lemma 4.2.5 Let A be a finitely presented algebra, then the following are equivalent:

(i) We have $\neg\neg\,\mathrm{Spec}(A)$.

(ii) For all f.p. algebras B, we have that:

$$(A \otimes B = 0) \to (B = 0)$$

Proof The second condition can be reformulated as:

$$\neg(\mathrm{Spec}(A) \times \mathrm{Spec}(B)) \to \neg\,\mathrm{Spec}(B)$$

so we get (ii) implies (i) by taking $B = R$, and (i) implies (ii) is just negation manipulation. □

This is a first step toward a flat f.p. algebra being faithfully flat if and only if its spectrum is non-empty. It shows one direction.

Next lemma should perhaps disappear?

Lemma 4.2.6 If $\rho : A \to B$ makes B a faithfully flat A-algebra, then ρ is injective.

Proof TODO □

4.3 Definition

Definition 4.3.1 The fppf pre-topology is the pre-topology consisting of $\mathrm{Spec}(A)$ for A faithfully flat.

Lemma 4.3.2 The fppf pre-topology is a topology.

Proof TODO □

4.4 The fppf topology is subcanonical

Lemma 4.4.1 Assume given a $A \to B$ a map of fp algebras, if $A \to B$ has a section then the sequence:

$$0 \to A \to B \to B \otimes_A B$$

where the last map send b to $b \otimes 1 - 1 \otimes b$ is exact.

Proof Form the hypothesis we get that $B = A \oplus A'$ and then we can compute using the distributivity of $\otimes$ over $\oplus$. □

Proposition 4.4.2 We have that R is an fppf sheaf.

Proof By unfolding the definition of sheaves, we just need to prove that for any faithfully flat algebra A we have an exact the sequence:

$$0 \to R \to A \to A \otimes_R A$$

where the last map send a to $a \otimes 1 - 1 \otimes a$. Since A is faithfully flat it is enough to prove that the induced sequence:

$$0 \to A \to A \otimes_R A \to A \otimes_R A \otimes_R A$$

is exact, but this sequence is isomorphic to the sequence:

$$0 \to A \to A \otimes_R A \to (A \otimes_R A) \otimes_A (A \otimes_R A)$$

where the last map sends $a \otimes b$ to $(a \otimes b) \otimes (1 \otimes 1) - (1 \otimes 1) \otimes (a \otimes b)$. This sequence is exact by lemma 4.4.1 since the map $A \otimes A \otimes_R A$ sending a to $a \otimes 1$ has a section sending $a \otimes b$ to ab. □