Synthetic Serre Duality

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(The following is a collection of notes on work in progress by (so far) Felix Cherubini and Fabian Endres.)

Contents

Introduction

Work on proving Serre-Duality, first for $\mathbb{P}^n$, then hopefully for many closed subsets thereof.

1 Coherent R-modules?

Classically, there is no abelian category of coherent $\mathcal{O}_X$-modules over a general base. Synthetically, the category of finitely presented modules over R does not have all kernels, since R is not coherent, so module bundles with values in finitely presented modules are not abelian.

This section is about finding a suitable replacement category.

Theorem 1.1
For an R-module M let $M^* :\equiv \mathrm{Hom}_{R\text{-Mod}}(M,R)$ be its dual. Then dualizing a finite co-presentation of M:

$$M \hookrightarrow R^n \to R^m$$

yields a finite presentation:

$$R^m \to R^n \twoheadrightarrow M^*$$

This induces an anti equivalence between finitely presented R-modules and finitely co-presented R-modules.

Proof [Che+23] □

Lemma 1.2 Let M, N be finitely presented and $f : M \to N$. Then there is an extension of this morphism to all presentations of M and N:

$$\begin{tikzcd} R^{m'} \arrow[r] \arrow[d, dotted, "\exists"] & R^m \arrow[r, two heads, "\pi_M"] \arrow[d, dotted, "\exists"] & M \arrow[d, "f"] \\ R^{n'} \arrow[r] & R^n \arrow[r, two heads, "\pi_N"] & N \end{tikzcd}$$

By dualizing, the analogous statement holds for finitely co-presented modules.

Proof By linear extension: For a standard basis vector $e_i : R^m$ we merely have $y : R^n$ such that $\pi_N(y) = f(\pi_M(e_i))$. By exactness, we also get $R^{m'} \to R^{n'}$ by linear extension. $\square$

Lemma 1.3 Let $f : M \to N$ be a map of finitely presented R-modules, then $\ker f$ is the cokernel of map between finitely copresented R-modules.

Proof By Lemma 1.2 we can assume f is induced by a square and we construct finitely copresented modules A, B like described below:

$$\begin{tikzcd}
A \arrow[r, "\pi_1"] \arrow[d, "l\times\mathrm{id}"] & R^{m'} \arrow[r, "t"] \arrow[d, "l"] & R^{n'} \arrow[d, "r"] \\
B \arrow[r, "\pi_1"] \arrow[d] & R^m \arrow[r, "b"] \arrow[d] & R^n \arrow[d] \\
K \arrow[r] & M \arrow[r, "f"] & N
\end{tikzcd}$$

$A := \{(z, y) : R^{m'} \times R^{n'} \mid blz = ry\}$, $B := \{(x, y) : R^m \times R^{n'} \mid bx = ry\}$. $\square$

Due to lack of a better name, we call the following modules coherent – the idea is that they might yield an analogue of the algebro-geometric notion of coherent sheaf of modules

Definition 1.4 Let M be an R-module, then M is called coherent, if it merely is the kernel of a homomorphism of finitely presented R-modules.

Theorem 1.5
Coherent modules are an abelian category.

Proof Using Lemma 1.3 and its dual. $\square$

Remark 1.6 Any coherent R-module is wqc.

Remark 1.7 For coherent R-modules M, N, the R-module $\mathrm{Hom}_{R\text{-Mod}}(M, N)$ is coherent.

Proof In [LQ15][Chapter IV, 4.12], "coherent" means something else, but their proof can still be applied twice to show the statement of the remark as follows.

First we use their proof to show that for finitely presented R-modules M, N the R-module $\mathrm{Hom}_{R\text{-Mod}}(M, N)$ is coherent. Morphisms $f : M \to N$ are presented by squares:

$$\begin{tikzcd}
R^m \arrow[r] \arrow[d, "\varphi"] & R^{m'} \arrow[d, "\psi"] \\
R^n \arrow[r] & R^{n'}
\end{tikzcd}$$

So we have a finite free module of pairs of morphisms (φ, ψ) and a submodule of pairs such that the square above commutes. This submodule S is the kernel of a linear map and therefore coherent. We have a surjection $\pi : S \to \mathrm{Hom}_{R\text{-Mod}}(M, N)$, so $\mathrm{Hom}_{R\text{-Mod}}(M, N)$ is coherent, if π is a cokernel of a map of coherent modules. This is the case, since there is a surjection onto $\ker \pi$ from the finite free R-module of linear maps which splitting the square:

$$\begin{tikzcd}
R^m \arrow[r] \arrow[d, "\varphi"] & R^{m'} \arrow[d, "\psi"] \arrow[dl, "s"] \\
R^n \arrow[r] & R^{n'}
\end{tikzcd}$$

Then we can reuse the same argument to show the statement of the remark. $\square$

Questions:

1. Is dualization an antiequivalence of coherent modules?
2. Are coherent modules closed under extension?
3. Are coherent modules closed under Hom?
4. Are coherent modules closed under $\_ \otimes \_$?
5. Is a locally coherent R-module coherent?
6. Are projective/proper/finite-scheme products of coherent modules coherent?
7. Are the cohomology groups of coherent module bundles on projective schemes coherent?

Answers:

1. Partial answer: Any coherent M can be written as a cokernel of finitely copresented modules:

$$A \to B \to M \to 0$$

By the properties of cokernels, the dual is exact:

$$0 \to M^\vee \to B^\vee \to A^\vee$$

with $B^\vee$ and $A^\vee$ finitely presented. So $M^\vee$ is coherent. So at the very least, dualization is a functor on coherent modules.

2. None so far
3. Yes, remark 1.7.
4. It could be possible to just use the result for finitely presented modules, by extending the extension properties above to bilinear maps.
5. None so far
6. Partial answer:
   (a) Let X be a finite scheme, i.e. X an affine scheme such that R^X is a finitely presented R-module. For any coherent R-module M, M^X is coherent: We can exponentiate the diagram witnessing coherence. Exponentiating is left exact and since M is wqc and X affine, it is also right exact.
   (b) Let $M : X \to R\text{-Mod}_{\text{coh}}$ for a finite scheme X. Then $(x : X) \to M_x$ would be a coherent R-module, if we have item 5: By local choice, we get a cover $D(f_1), \dots, D(f_n)$ of X, such that coherence of M is witnessed by a diagram $M \hookrightarrow N \to L$ with N, L finitely presented. By [CCH24][Theorem 7.2.3], $(x : D(f_i)) \to N_x$ and $(x : D(f_i)) \to L_x$ are finitely presented $R^{D(f_i)}$-modules and therefore finitely presented R-modules and the dependent product is a left exact functor.
7. None so far. It seems a bit much to ask, but it would be really great for Serre-Duality, so it should be worth it to try to adapt e.g. [Vak, p. 19.1.3].

2 Projective Space

We use notation from [CCH24] and [Che+24].

Definition 2.1 $\mathbb{P}^n$ may be defined as one of the following equivalent types:
(i) The type of lines[1] through the origin in R^{n+1}.
(ii) The set-quotient of $R^{n+1} \setminus \{0\}$ by the relation $x \sim y$ iff $x = \lambda y$ for some $\lambda : R^\times$.

[1] Submodules $M \subseteq R^{n+1}$ with $\|M = R^1\|$

(iii) The homotopy quotient:
$$\sum_{l:K(R^\times,1)} l^{n+1} \setminus \{0\}$$

Lemma 2.2 Let $U \subseteq \mathbb{P}^n$. The following are equivalent:

(i) U is open.

(ii) $U = D(P_0, \dots, P_l)$ for $P_i : R[X_0, \dots, X_n]_d$.

(iii) There are $N : \mathbb{N}$ and $s_0, \dots, s_l : \prod \mathcal{O}(1)^{\otimes N}$ such that
$$U = D(s_0, \dots, s_l) :\equiv \sum_{x:X} s_0(x) \neq 0 \vee \cdots \vee s_l(x) \neq 0.$$

The equivalence between the last two statements follows from [BCW23][Theorem 8.3]. The following is a direct consequence of Definition 2.1 (iii):

Remark 2.3 The type of maps $X \to \mathbb{P}^n$ is equivalent to the type
$$(\mathcal{L} : X \to K(R^\times, 1)) \times (s_0, \dots, s_n : (x : X) \to \mathcal{L}_x) \to (x : X) \to \exists_i s_i(x) \neq 0$$
i.e. the type of line bundles on X that come with $n+1$ sections that do not simultaneously vanish.

Definition 2.4 A line bundle $\mathcal{L} : X \to K(R^\times, 1)$ is called *very ample* if one of the following equivalent statement holds:

(i) There is a closed embedding $i : X \to \mathbb{P}^n$ and $\mathcal{L} = i_*\mathcal{O}(1)$.

(ii) There are non-simultaneously vanishing sections $s_0, \dots, s_n$ of $\mathcal{L}$ such that $[s_0, \dots, s_n] : X \to \mathbb{P}^n$ is a closed embedding.

Proof All we have to do is to show that if (ii) holds, for $i :\equiv [s_0, \dots, s_n]$, there is an equality $\mathcal{L} = i_*\mathcal{O}(1)$. Let $x : X$ and note first that $\mathcal{O}(-1)_x$ is generated by $(s_0, \dots, s_n)^T$. Let $\varphi : R^1 \to \mathcal{L}_x$ be an isomorphism of R-modules. Then there are $\lambda_0, \dots, \lambda_n$ such that $\sum \lambda_i s_i(x) = \varphi(1)$, which lets us define:
$$\begin{array}{ccc} \mathcal{L}_x & \longrightarrow & \mathcal{O}(1) = \mathrm{Hom}_R(\mathcal{O}(-1), R) \\ \\ \lambda \cdot \varphi(1) & \longmapsto & \sum \lambda_i \varphi^{-1}(s_i(x)) \end{array}$$
which is an isomorphism independent of the choice of φ. □

This is a variant of [Vak][Theorem 17.6.2] (which might hold):

Definition 2.5 Let X be a proper scheme. A line bundle $\mathcal{L} : X \to K(R^\times, 1)$ is *ample* if one of the following equivalent statements holds:

(i) There is $N > 0$ such that $\mathcal{L}^{\otimes N}$ is very ample.

(ii) Any open subset $U \subseteq X$ is of the form $U = D(s_0, \dots, s_n)$, for some $s_i : \mathcal{L}^{\otimes l_i}(X)$ with $l_i > 0$.

(iii) Any open subset $U \subseteq X$ is of the form $U = D(s_0, \dots, s_n)$, for some $N > 0$ and $s_i : \mathcal{L}^{\otimes N}(X)$.

(iv) For any finite type, weakly quasi-coherent module bundle $\mathcal{F} : X \to R\text{-Mod}_{\mathrm{wqc,fg}}$, there is $N > 0$, $j : \mathbb{N}$ and a (pointwise) surjection $\mathcal{O}^{\oplus j} \to \mathcal{F} \otimes \mathcal{L}^{\otimes N}$.

We expect the following to be true:

Lemma 2.6 Let $\mathcal{F} : \mathbb{P}^n \to R\text{-Mod}_{\mathrm{fg}}$ be a finitely generated R-module bundle. Then there merely are n, q and a surjection
$$\mathcal{O}(q)^{\oplus n} \to \mathcal{F}.$$
Furthermore, if $\mathcal{F}$ is finitely presented, there are n', n, q', q and an exact sequence:
$$\mathcal{O}(q')^{\oplus n'} \to \mathcal{O}(q)^{\oplus n} \to \mathcal{F} \to 0.$$

Proof TODO. [LQ15, Chapter IV, 1.0 Lemma] for the second part, to see that the kernel of a surjection from a finite free module to a finitely presented module is finitely generated. □

3 Cohomology and Consequences

We use the synthetic algebro-geometric notions of [CCH24] and recollect some results of [BCW23]. This entails that we use a pullback f_* and push-forward f^* which arguably deviates somewhat from the usual notion and is more of a pointwise nature.

Definition 3.1 (a) Let $A, B : X \to R\text{-Mod}$. For subsets $U \subseteq X$, we write $A(U) :\equiv (x : U) \to A_x$. Similarly, if for each $x : X$ we have a morphisms $f_x : A_x \to B_x$, we write $f : A \to B$ and $m_U :\equiv (s : A(U)) \mapsto ((x : U) \mapsto f_x(s(x)))$.

(b) Let $A, B, C : X \to R\text{-Mod}$. If for all $x : X$ we have a sequence

$$A_x \xrightarrow{m_x} B_x \xrightarrow{e_x} C_x$$

we write $A \to B \to C$ and call this datum a sequence. The sequence is called *(pointwise) exact* if for all $x : X$ the sequence above is exact. The sequence is called *locally exact*, if e is surjective and there is an open affine cover such that m_U is injective.

Note that pointwise exactness is a strictly stronger requirement than local exactness.

Theorem 3.2

(a) Let $A, B, C : X \to \text{Ab}$ and

$$0 \to A \to B \to C \to 0$$

be a (pointwise) exact sequence, then there is a long exact sequence of cohomology groups:

$$\begin{tikzcd} & & \dots \arrow[r] & H^{n-1}(X,C) \arrow[dll] \\ & H^n(X,A) \arrow[r] & H^n(X,B) \arrow[r] & H^n(X,C) \arrow[dll] \\ & H^{n+1}(X,A) \arrow[r] & H^{n+1}(X,B) \arrow[r] & \dots \end{tikzcd}$$

(b) Let $A, B, C : X \to R\text{-Mod}_{\text{wqc}}$ and

$$0 \to A \to B \to C \to 0$$

be a locally exact sequence, then there is an induced long exact sequence like above.

Theorem 3.3

Let X be a separated scheme with open affine cover $\{U\}$ and $M : X \to R\text{-Mod}_{\text{wqc}}$ a bundle of weakly quasi-coherent R-modules. The natural isomorphism $H^0(X, M) \cong \check{H}^0(\{U\}, M)$ extends to an isomorphism of ∂-functors, i.e. it extends to a natural isomorphism

$$H^k(X, M) \cong \check{H}^k(\{U\}, M), \text{ for all } k \geq 0,$$

compatible with long exact cohomology sequences.

There is also the helpful lemma:

Lemma 3.4 Let $f : Y \to X$ and $A : Y \to \text{Ab}$ be such that $H^l(\text{fib}_f(x), \pi_1^* A) = 0$ for all $0 < l \leq n$, then

$$H^n(Y, A) = H^n(X, f_* A).$$

In particular, if $i : C \to \mathbb{P}^n$ is closed and $M : \mathbb{P}^n \to R\text{-Mod}_{wqc}$, then $H^n(C, i_* M) = H^n(\mathbb{P}^n, M)$.

With these results, the proof of the following carries over[2] from [Har77, Chapter III]:

Theorem 3.5

(a) For all $n : \mathbb{N}$, $d : \mathbb{Z}$, there are isomorphisms $R[X_0, \dots, X_n]_d \to H^0(\mathbb{P}^n, \mathcal{O}(d))$ of R-modules, inducing an isomorphism $R[X_0, \dots, X_n] \to \bigoplus_{d:\mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{O}(d))$ of graded $R[X_0, \dots, X_n]$-modules.

[2] A full proof in the synthetic setting is given in [BCW23].

(b) $H^n(\mathbb{P}^n, \mathcal{O}(-n-1)) = R$ is free of rank 1 and $H^n(\mathbb{P}^n, \mathcal{O}(d)) = 0$ for $d > -n-1$.

(c) The canonical map given by tensoring
$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \times H^n(\mathbb{P}^n, \mathcal{O}(-d-n-1)) \to R$$
is a perfect pairing of finite free R-modules for all $d : \mathbb{Z}$.

(d) $H^i(\mathbb{P}^n, \mathcal{O}(d)) = 0$ for $i \in \{1, \dots, n-1\}$ and all $d : \mathbb{Z}$.

Following [Vak, Chapter 19].

4 Duality

Lemma 4.1 Let X be a type and $\mathcal{L} : X \to \mathsf{Line}$ a line bundle. For all $\mathcal{F} : X \to R\text{-Mod}$ we have an isomorphism of R-module bundles:
$$\mathit{Hom}_{R\text{-Mod}(X)}(\mathcal{L}, \mathcal{F}) \to \mathcal{L}^\vee \otimes \mathcal{F}.$$
(where $\mathit{Hom}_{R\text{-Mod}(X)}(\mathcal{L}, \mathcal{F}) = (x : X) \mapsto \mathrm{Hom}_{R\text{-Mod}}(\mathcal{L}_x, \mathcal{F}_x))$ In particular, we have
$$\mathrm{Hom}_{R\text{-Mod}(X)}(\mathcal{L}, \mathcal{F}) \simeq H^0(X, \mathcal{L}^\vee \otimes \mathcal{F}).$$

Proof Let $x : X$ and choose an isomorphism of R-modules $\mu : \mathcal{L}_x \to R$. Note first, that we have an isomorphism
$$\psi : \mathrm{Hom}_{R\text{-Mod}}(R, \mathcal{F}_x) \to R^\vee \otimes \mathcal{F}_x$$
given by mapping $\varphi : \mathrm{Hom}_{R\text{-Mod}}(R, \mathcal{F}_x)$ to $\mathrm{id}_R \otimes \varphi(1)$. So
$$\mu^\vee \otimes \mathrm{id}_R \circ \psi \circ (\_ \circ \mu^{-1}) : \mathrm{Hom}_{R\text{-Mod}}(\mathcal{L}_x, \mathcal{F}_x) \to {\mathcal{L}_x}^\vee \otimes \mathcal{F}_x$$
is a map of the desired type and by elemtary calculation using linearity it is independent of the choice of μ. □

(We are not sure, whether the following theorem is true in the stated generality) We write $\omega = \omega_X$ for the canonical line bundle over X, which we assume (remains to be proven) to be isomorphic to $\mathcal{O}(-1-n)$ for $X = \mathbb{P}^n$.

Theorem 4.2 (Duality for $\mathbb{P}^n$)
Let $\mathcal{F} : \mathbb{P}^n \to R\text{-Mod}_{\mathrm{fp}}$ be a finitely presented R-module bundle.

(a) $H^n(\mathbb{P}^n, \omega) = R$.

(b) There is a pairing (natural in $\mathcal{F}$?)
$$\mathrm{Hom}_{R\text{-Mod}(X)}(\mathcal{F}, \omega) \times H^n(X, \mathcal{F}) \to H^n(X, \omega) \simeq R$$
inducing an isomorphism for $\mathrm{Hom}_{R\text{-Mod}(X)}(\mathcal{F}, \omega) \to H^n(X, \mathcal{F})^\vee$.

(c) (Hartshorne asks for a field, to make the dualization below exact.) For every $i \geq 0$ there is a natural functorial isomorphism
$$\mathrm{Ext}^i(\mathcal{F}, \omega) \to H^{n-i}(X, \mathcal{F})^\vee$$

Proof (UNCLEAR), following [Har77, Ch. III, 7.1].

(a) This follow from Theorem 3.5 (b)

(b) The pairing is given by letting $\varphi : \mathrm{Hom}_{R\text{-Mod}(X)}(\mathcal{F}, \omega)$ induce a $\varphi^* : H^n(X, \mathcal{F}) \to H^n(X, \omega)$. In the case $\mathcal{F} \simeq \mathcal{O}(q)$, we can use Lemma 4.1 to compute $\mathrm{Hom}_{R\text{-Mod}(X)}(\mathcal{O}(q), \omega) \simeq H^0(\mathbb{P}^n, \omega \otimes \mathcal{O}(-q))$. So in this case, we can conclude by Theorem 3.5 (c).

By Lemma 2.6, we get an exact sequence:
$$\mathcal{O}(p)^{\oplus n} \to \mathcal{O}(p')^{\oplus n'} \to \mathcal{F} \to 0$$
and the functors $\mathrm{Hom}_{R\text{-Mod}(X)}(\_, \omega)$ and $H^n(X, \_)^\vee$ both map right exact sequences to left exact sequences. So we can conclude with the 5-lemma.

(c) Idea: Prove that projective schemes have coherent $H^i(X, \mathcal{F})$ for all i and show that dualizing coherent modules is exact. Then the proof in Hartshorne might go through. □

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