Random Facts in Synthetic Algebraic Geometry

???

April 26, 2024

Abstract

The following is a collection of results in synthetic algebraic geometry, that didn't find a place anywhere else so far.

Contents

1 Translations of Classical Proofs

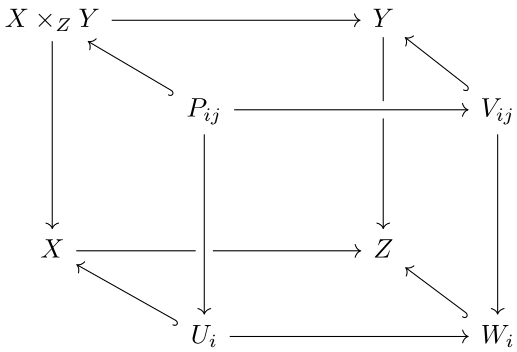
This is a proof that pullbacks of schemes are schemes, which is analogous to what could be found in a textbook.

Theorem 1.0.1 (using ??, ??, ??)
Let

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

be schemes, then the *pullback* $X \times_Z Y$ is also a scheme.

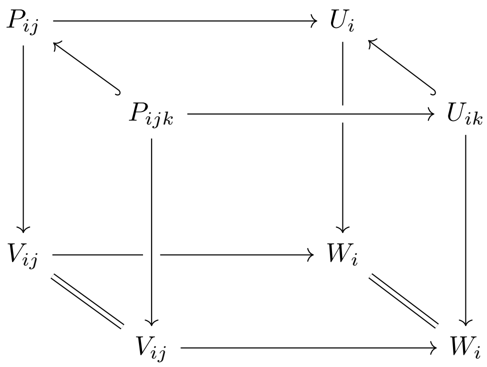
Proof (alternative proof of theorem 1.0.1) Let $W_1, \dots, W_n$ be a finite affine cover of Z. The preimages of W_i under f and g are open and covered by fintely many affine open U_{ik} and V_{ij} by **??**. This leads to the following diagram:



where the front and bottom square are pullbacks by definition. By pullback-pasting, the top is also a pullback, so all diagonal maps are embeddings.

P_{ij} is open, since it is a preimage of V_{ij} (**??**), which is open in Y by **??**. It remains to show, that the P_{ij} cover $X \times_Z Y$ and that P_{ij} is a scheme. Let $x : X \times_Z Y$. For the image w of x in W, there merely is an i such that w is in W_i. The image of x in V_i merely lies in some V_{ij}, so x is in P_{ij}.

We proceed by showing that P_{ij} is a scheme. Let U_{ik} be a part of the finite affine cover of U_i. We repeat part of what we just did:



So by **??**, P_{ijk} is affine. Repetition of the above shows, that the P_{ijk} are open and cover P_{ij}. ☐

2 Homogeneity

Definition 2.0.1 A *graded* R-algebra S is an R-algebra S together with the datum of a direct sum decomposition

$$S = \oplus_{n \in \mathbb{Z}} S_n$$

as an R-module such that $S_k \cdot S_\ell \subseteq S_{k+l}$ for every k, $\ell \in \mathbb{Z}$. We identify each S_n for $n \in \mathbb{Z}$ with its image in S. The elements of S_n are called *homogeneous* of degree n.

Remark 2.0.2 The datum of a grading of an R-algebra S thus gives us an essentially finite decomposition[1] $u = \sum_{n \in \mathbb{Z}} u_n$ for every $u \in S$ where each u_n is homogeneous of degree n. Furthermore, the decomposition of a homogeneous element u is just $u = u$.

Proposition 2.0.3 Let S be a graded R-algebra and let $X := \operatorname{Spec} S$. For each $t \in R^\times$ and each $p \in X$ we define

$$t \cdot p := \left(u \mapsto \sum_{n \in \mathbb{Z}} p(u_n) t^n \right) \in \operatorname{Spec} S.$$

This defines an operation of the multiplicative group $R^\times$ on X.

Proof That $1 \cdot p = p$ for every $p \in X$ follows from $p(u) = \sum_{n \in \mathbb{Z}} p(u_n) \cdot 1^n$. That $r \cdot (r' \cdot p) = (r \cdot r') \cdot p$ follows from the homogeneity of every u_n. ☐

Theorem 2.0.4
The construction in proposition 2.0.3 yields an identification between the type of graded R-algebras and the type of affine schemes together with an action of the multiplicative group $R^\times$.

[1]That means there merely are $i, j : \mathbb{Z}$ such that $u_n = 0$ if $n < i$ or $n > j$.

Proof We give the converse construction. Let $X = \operatorname{Spec} S$ be an affine scheme together with the datum $R^\times \times X \to X$ of an action of the multiplicative group $R^\times$. By synthetic quasi-coherence, the function $R^\times \times X \to X$ yields a homomorphism

$$\alpha(t)\colon S \to S[t, t^{-1}] = S \otimes R[t, t^{-1}], u \mapsto \sum_{n \in Z} u_n t^n$$

of R-algebras where the sum on the right hand side is essentially finite. That $1 \cdot p = p$ for all $p \in X$ is equivalent to $\alpha(1) = \mathrm{id}_S$, which, in turn, yields $u = \sum_{n \in \mathbb{Z}} u_n$. That $t \cdot (t' \cdot p) = (t \cdot t') \cdot p$ is equivalent to $(\alpha(t) \otimes \mathrm{id}_{R[t,t^{-1}]}) \circ \alpha(t') = \alpha(t \cdot t')$, from which $\alpha(t)(u_n) = u_n t^n$ follows. ☐

Remark 2.0.5 Let S be a graded R-algebra. The action of $R^\times$ on $\operatorname{Spec} S$ induces a natural action of $R^\times$ on the R-algebra of functions on $\operatorname{Spec} S$ given by

$$R^{\operatorname{Spec} S} \times R^\times \to R^{\operatorname{Spec} S}, (f, t) \mapsto (p \mapsto f(t \cdot p)).$$

Under the identification $R^{\operatorname{Spec} S} = S$ given by synthetic quasicoherence, this gives the action

$$S \times R^\times \to S, (u, t) \mapsto \sum_{n \in \mathbb{Z}} u_n t^n$$

of $R^\times$ on the R-algebra S.

Example 2.0.6 Let V be a finitely presented R-module. The symmetric algebra $\mathrm{Sym}^* V$ of V over R is naturally graded. In particular, the affine scheme

$$V^\vee := \operatorname{Spec} \mathrm{Sym}^* V$$

carries a natural action by the multiplicative group $R^\times$.

Remark 2.0.7 We write $V^\vee$ as

$$\operatorname{Spec} \mathrm{Sym}^* V = \mathrm{Hom}_{R\text{-Alg}}(\mathrm{Sym}^* V, R) = \mathrm{Hom}_{R\text{-Mod}}(V, R)$$

by the universal property of $\mathrm{Sym}^* V$. In particular, $V^\vee$ carries the structure of a (finitely copresented) R-module. Moreover, the natural action of $R^\times$ on the left hand side corresponds to scalar multiplication on the right hand side.

From the above, we can deduce that V is reflexive:

Theorem 2.0.8
Let V be a finitely presented R-module. Set

$$V^{\vee\vee} := \mathrm{Hom}_{R\text{-Mod}}(V^\vee, R).$$

The natural map

$$V \to V^{\vee\vee}, f \mapsto (p \mapsto p(f))$$

is an isomorphism of R-modules. In particular, V is reflexive and the dual of every finitely copresented R-module (which is always the dual of a finitely presented R-module) is finitely presented.

Proof The identification

$$\mathrm{Sym}^* V \to R^{\operatorname{Spec} \mathrm{Sym}^* V} = R^{V^\vee}$$

is an identity of R-algebras with an $R^\times$-action. In particular, the homogeneous elements of degree 1 on the left hand side correspond to the homogeneous elements of degree 1 on the right hand side. This yields an isomorphism

$$\phi\colon V \to \mathrm{Hom}_{R^\times}(V^\vee, R) := \{u\colon V^\vee \to R \mid \forall t \in R^\times \forall v \in V\colon u(vt) = u(v)t\}, v \mapsto (p \mapsto p(v))$$

of R-modules. As the image of ϕ lies inside $V^{\vee\vee} \subseteq \mathrm{Hom}_{R^\times}(V^\vee, R)$ we actually have $V^{\vee\vee} = \mathrm{Hom}_{R^\times}(V^\vee, R)$ and the theorem is proven. ☐

Remark 2.0.9 It follows from the above that for a finitely copresented R-module $V^\vee$, a (1-)homogenous map $V^\vee \to R$ is already R-linear (principle of microlinearity).

3 Serre's affineness theorem

Ingo Blechschmidt and David Wärn proved the following analogue of Serre's theorem on affineness

Theorem 3.0.1
Let X be a scheme such that $H^1(X, I) = 0$, for all $I : X \to R\text{-Mod}_{\text{wqc}}$, then X is affine.

Here is a consequence:

Corollary 3.0.2 Let X be an affine scheme and $Y : X \to \text{Sch}_{\text{qc}}$, such that each Y_x is affine. Then $(x : X) \times Y_x$ is affine.

Proof Let $M : (x : X) \times Y_x \to R\text{-Mod}_{\text{wqc}}$. Then (explanations for the steps below):

$$\begin{aligned} H^1((x : X) \times Y_x, M) &= \|((x, y_x) : (x : X) \times Y_x) \to K(M_{(x,y_x)}, 1))\|_{\text{set}} \\ &= \|(x : X) \to ((y_x : Y_x) \to K(M_{(x,y_x)}, 1))\|_{\text{set}} \\ &= \|(x : X) \to K((y_x : Y_x) \to M_{(x,y_x)}, 1)\|_{\text{set}} \\ &= 0 \end{aligned}$$

□

The first step, after expanding the definition, is just currying. To commute the Eilenberg-MacLane space with the dependent function type, we use that Y_x is affine and therefore the type $(y_x : Y_x) \to K(M_{(x,y_x)}, 1)$ is connected. It is a delooping of $(y_x : Y_x) \to M_{(x,y_x)}$, so by connectedness, it must be equivalent to $K((y_x : Y_x) \to M_{(x,y_x)}, 1)$. The last step uses that X is affine, and $(y_x : Y_x) \to M_{(x,y_x)}$, as a scheme-indexed product of weakly quasi-coherent modules, is again weakly quasi-coherent.

4 Line bundles and divisors

4.1 Regular sections and regular closed subschemes

In classical algebraic geometry, there is the concept of a *generic section* of a line bundle. Informally, the generic sections have the smallest possible vanishing set. The following definition corresponds to this notion:

Definition 4.1.1 Let X be a type and $\mathcal{L} : X \to R\text{-Mod}$ a line bundle. A section

$$s : \prod_{x:X} \mathcal{L}_x$$

is *regular*, there merely is a trivializing affine cover $U_1 = \operatorname{Spec} A_1, \ldots, U_n = \operatorname{Spec} A_n$ of $\mathcal{L}$, such that each trivialized restriction

$$s_i : \operatorname{Spec} A_i \to R$$

is a regular element (**??**) of $(\operatorname{Spec} A_i \to R) = A_i$.

Lemma 4.1.2 Let $s : \operatorname{Spec} A \to R$. s being regular is Zariski-local, i.e. for all Zariski-covers $U_1, \ldots, U_n$ of $\operatorname{Spec} A$, s is regular, if and only if it is regular on all U_i.

Proof It is enough to check this for a localization at $f : A$. Let

$$\frac{s}{1} \cdot \frac{g}{f^k} = 0.$$

then $f^l sg = 0$, which implies $f^l g = 0$ by regularity of s and therefore $\frac{g}{f^l} = 0$. □

Proposition 4.1.3 The choice of trivializing cover in definition 4.1.1 is irrelevant.

Proof By lemma 4.1.2. □

From a line bundle together with a regular section, we can produce a closed subtype of a special kind:

Definition 4.1.4 Let X be a scheme. A *regular closed subtype* of X is a closed subtype $C : X \to \text{Prop}$, such that there merely is an affine open cover $U_1 = \operatorname{Spec} A_1, \ldots, U_n = \operatorname{Spec} A_n$, and $C \cap U_i$ is $V(f_i)$ for a regular $f_i : A_i$.

Lemma 4.1.5 Let $f, g : A$, f be regular and $V(f) = V(g)$, then g is regular and there is a unique unit $\alpha : A^\times$, such that $\alpha f = g$.

Proof $V(f) = V(g)$ implies there are $\alpha, \beta : A$ such that $\alpha f = g$ and $\beta g = f$. But then: $f = \beta g = \beta\alpha f$. So by regularity of f, $\beta\alpha = 1$. By **??**, units are regular and products of regular elements are regular, so g is regular. Uniqueness of α follows from regularity. □

Theorem 4.1.6 (using ??)
Let X be a scheme. For any regular closed subscheme C, there is a line bundle with regular section $(\mathcal{L}, s)$ on X, such that $C = V(s)$.

Proof Let $U_1 = \operatorname{Spec} A_1, \ldots, U_n = \operatorname{Spec} A_n$ be a cover by standard affine opens such that we have regular f_i with $C \cap U_i = V(f_i)$. We define $\mathcal{L}$ to be the trivial line bundle $\_ \mapsto R$ on each U_i and by giving automorphisms on the intersections $U_i \cap U_j :\equiv U_{ij} = \operatorname{Spec} A_{ij}$. On U_{ij}, C is given by $V(\frac{f_i}{1})$ and $V(\frac{f_j}{1})$ which are both regular. Therefore, there is a unit $\alpha : A_{ij}^\times$ such that $\alpha\frac{f_i}{1} = \frac{f_j}{1}$, which we can also view as a map $U_{ij} \to R^\times$ and since $R^\times$ is equivalent to the automorphism group of R as an R-module, this provides the identetification we need to construct $\mathcal{L}$. Under the identification, the local regular sections are identified, so we get a global section s of $\mathcal{L}$, which is locally regular. □

5 Segre Embedding

Maybe this should be moved to the draft on proper schemes. This section just repeats classical knowledge which happens to work synthetically.

Definition 5.0.1 (a) A *projective scheme* is type X such that there is a closed subset of some $\mathbb{P}^n$ equivalent to X.

(b) A *quasi-projective scheme* is type X such that there is an open subset of a closed subset of some $\mathbb{P}^n$ equivalent to X.

We write $[x]$ for the point in $\mathbb{P}^n$ given by a vector $x : R^{n+1}$.

Definition 5.0.2 The *Segre-Embedding* is the map $s : \mathbb{P}^s \times \mathbb{P}^t \to \mathbb{P}^{s \cdot t}$ given by

$$([x_0 : \cdots : x_s], [y_0 : \cdots : y_t]) \mapsto [(x_i \cdot y_j)_{i,j}]$$

Proposition 5.0.3 The Segre-Embedding is a closed embedding.

Proof First, let $[x], [x'] : \mathbb{P}^s$ and $[y], [y'] : \mathbb{P}^t$ such that there is a $\lambda : R^\times$ with $x_i y_j = \lambda x'_i y'_j$ for all i, j. There is a l, ki such that $x_l \neq 0$ and $y_k \neq 0$. By the condition above, this implies $x'_l \neq 0$ and $y'_k \neq 0$. Then we have

$$x_i = \lambda \cdot \frac{x'_i y'_k}{y_k} = \frac{x_l y_k}{x'_l y'_k} \cdot \frac{x'_i y'_k}{y_k} = \frac{x_l}{x'_l} \cdot x'_i.$$

This shows the Segre-Embedding is an embedding.

A point $[z] : \mathbb{P}^{s \cdot t}$ is in the image of the Segre-Embedding, if and only if the equation $z_{ij} \cdot z_{kl} = z_{il} \cdot z_{kj}$ holds: We have an invertible z_{ij} and can define $x_l :\equiv z_{lj} \cdot z_{ij}^{-1}$ and $y_k :\equiv z_{ik} \cdot z_{ij}^{-1}$. Then $x_l \cdot y_k = z_{lj} \cdot z_{ij}^{-1} \cdot z_{ik} \cdot z_{ij}^{-1} = z_{lk} \cdot z_{ij}^{-1}$, which shows that $[(z_{lk} \cdot z_{ij}^{-1})_{lk}]$ is in the image of the Segre-Embedding and therefore $[z]$. □

Theorem 5.0.4
The type of (quasi-)projective schemes is closed under products of schemes.

6 Blow-up of Projective Space

(This does not correspond to the usual Blow-up and should be thought of as a blow-up of a system of equations.)

Definition 6.0.1 Let $V = V(P_1, \dots, P_l) \subseteq \mathbb{P}^n$ be a closed subset given by homogenous polynomials $P_1, \dots, P_l$ of the same degree. Then the closed subset given by

$$\begin{aligned}\mathrm{Bl}_V : &\mathbb{P}^n \times \mathbb{P}^{l-1} \to \mathrm{Prop} \\ &([x],[y]) \mapsto \bigwedge_{i,j} (P_i(x) \cdot y_j = P_j(x) \cdot y_i)\end{aligned}$$

is called the *blow-up* of $\mathbb{P}^n$ at V. There is a projection $\pi_V : \mathrm{Bl}_V \to \mathbb{P}^n$.

Proposition 6.0.2 Let $V = V(P_1, \dots, P_l) \subseteq \mathbb{P}^n$ be a closed subset given by homogenous polynomials $P_1, \dots, P_l$ of the same degree.
(a) Bl_V is a projective scheme.

(b) For $U :\equiv D(P_1, \dots, P_l)$ we have $\pi_V^{-1}(U) = U$.

Proof (a) By Segre-Embedding.

(b) By definition of Bl_V, the vectors y and $(P_1(x), \dots, P_l(x))$ are linearly dependent. So $[x] : \mathbb{P}^n$ is in U if and only if $(P_1(x), \dots, P_l(x))$ has an invertible entry. But in that case, $[y]$ is uniquely determined, so the corestriction of π_V to U is an equivalence. □

Definition 6.0.3 Let $V = V(P_1, \dots, P_l) \subseteq \mathbb{P}^n$ be a closed subset given by homogenous polynomials $P_1, \dots, P_l$ of the same degree. Then $\mathcal{L}_v([x],[y]) :\equiv R\langle y \rangle$ defines a line bundle on Bl_V.

Proposition 6.0.4 In the same sitaution there is a pointwise at most rank 1 submodule $\mathcal{I}_V \subseteq \mathcal{L}_V$ on Bl_V such that $\mathcal{I}_{V,x} \neq 0$ implies $D(P_1, \dots, P_l)(\pi_V(x))$.

Proof (Sketch) $\mathcal{I}_{V,x}$ is generated by the vector $(P_1, \dots, P_l)$. □

This means that we transformed the quasi-projective U into an equivalent quasi-projective $\pi_V^{-1}(U) \subseteq \mathrm{Bl}_V$ which is locally a standard-open.

7 External justification of axioms

This is an unfinished justification of the axioms of SAG, using Kripke-Joyal Semantics. This was once a part of [CCH23].

7.1 Justification of ??

Lemma 7.1.1 Let (C, J) be a site, where the Grothendieck topology J is subcanonical. Let

$$f : E \twoheadrightarrow \mathrm{y}(c)$$

be an epimorphism in $\mathrm{Sh}(C, J)$ with representable codomain. Then there is a J-cover $(c_i \to c)_{i \in I}$ of c such that for every i, the pullback of f along $\mathrm{y}(c_i) \to \mathrm{y}(c)$ is a split epimorphism.

$$\begin{array}{ccc} E_i & \longrightarrow & E \\ \downarrow{\scriptstyle f_i} \; (\lrcorner) & & \downarrow{\scriptstyle f} \\ \mathrm{y}(c_i) & \longrightarrow & \mathrm{y}(c) \end{array}$$

Proof By the Yoneda lemma, an epimorphism $E \twoheadrightarrow \mathrm{y}(c)$ is split if and only if the particular element $\mathrm{id}_c \in \mathrm{y}(c)(c)$ is in the image of the map $E(c) \to \mathrm{y}(c)(c)$. Applying the usual characterization of epimorphisms of sheaves [MM12, Corollary III.7.5] to the element $\mathrm{id}_c \in \mathrm{y}(c)(c)$ shows that there is a J-cover $(c_i \xrightarrow{g_i} c)_{i \in I}$ such that for every $i \in I$, there is some $e_i \in E(c_i)$ with $f_{c_i}(e_i) = g_i \in \mathrm{y}(c)(c_i)$. But this means that id_{c_i} is in the image of $(f_i)_{c_i} : E_i(c_i) \to \mathrm{y}(c_i)(c_i)$, as we can see by evaluating the pullback diagram at c_i. So f_i is a split epimorphism. □

Let us formulate a version of the axiom **??** in infinitary first-order logic extended with unbounded quantification over objects/sorts ($\exists A.\varphi$, $\forall A.\varphi$) and quantification over functions ($\exists f : A \to B.\varphi$, $\forall f : A \to B.\varphi$) as in Shulmans stack semantics [Shu10, Section 7].

We also use the syntax $\{x : A \mid \varphi(x)\}$ for bounded set comprehension, but this can be translated away. TODO

$$
\begin{aligned}
\varphi_0 &:\equiv \bigwedge_{n,m\in\mathbb{N}} \forall r_1,\ldots,r_m : R[X_1,\ldots,X_n].\ \varphi_1 \\
\operatorname{Spec} A &:\equiv \{x : R^n \mid \operatorname{ev}(r_1,x) = \cdots = \operatorname{ev}(r_1,x) = 0\} \\
\varphi_1 &:\equiv \forall E.\ \forall \pi : E \to \operatorname{Spec} A.\ ((\forall x : \operatorname{Spec} A.\ \exists e : E.\ \pi(e) = x) \Rightarrow \varphi_2) \\
\varphi_2 &:\equiv \bigvee_{k\in\mathbb{N}} \exists f_1,\ldots,f_k : R[X_1,\ldots,X_n].\ f_1 + \cdots + f_k = 1 \wedge \varphi_3 \\
D(f_i) &:\equiv \{x : \operatorname{Spec} A \mid \exists y.\ \operatorname{ev}(f_i,x)y = 1\} \\
\varphi_3 &:\equiv \bigwedge_{i=1}^{k} \exists s : D(f_i) \to E.\ \forall x : D(f_i).\ \pi(s(x)) = x
\end{aligned}
$$

8 Group schemes are not smooth

The first example is very classical, we just give it because it is interesting.

Lemma 8.0.1 Assume $2 = 0$. Then the sub-group:

$$\mu_2 = \operatorname{Spec}(R[X]/X^2 - 1) \subset \mathbb{A}^\times$$

is not smooth.

Proof Since $2 = 0$, for all $x : R$ we have $x^2 = 1$ iff $(X^2 - 1)$. So as a scheme we have:

$$\mu_2 = \operatorname{Spec}(R[X]/(X-1)^2) = \operatorname{Spec}(R[X]/X^2) = \mathbb{D}(1)$$

which is not smooth. □

Proposition 8.0.2 Not all group schemes are smooth (without any assumption on the characteristic).

Proof For any closed proposition P consider the closed subgroup:

$$1 + P \subset \mathbb{Z}/2\mathbb{Z}$$

sending 1 to the the unit 0 and P to 1. If the group is smooth then so is P, which would then be decidable. □

9 Hilbert's Nullstellensatz

In this section we prove that the Jacobson radical of any f.p. algebra A is equal to its nilradical. The idea for this result as well as the proof for $A = R$ was given to me (Hugo Moeneclaey) by Max Zeuner.

Definition 9.0.1 Let A be an R-algebra, then we define the nilradical $\operatorname{Nil}(A)$ of A as the ideal of nilpotents in A.

Definition 9.0.2 Let A be an R-algebra, then we define the Jacobson radical $\operatorname{Jac}(A)$ of A as the ideal of $a : A$ such that forall $b : A$ we have $1 - ba$ invertible.

Classically, I think this is equivalent to the intersection of all maximal ideal.

Lemma 9.0.3 For any R-algebra A we have:

$$\operatorname{Nil}(A) \subset \operatorname{Jac}(A)$$

Proof Because $1 + x$ is invertible for x nilpotent. □

Lemma 9.0.4 We have:

$$\mathrm{Nil}(R) = \mathrm{Jac}(R)$$

Proof We have that:

$$\begin{aligned} &\forall (y : R).\ 1 - xy \text{ inv} \\ \Leftrightarrow\ &\forall (y : R).\ \neg(xy = 1) \\ \Leftrightarrow\ &\neg(\exists y : R.\ xy = 1) \\ \Leftrightarrow\ &\neg\neg(x = 0) \\ \Leftrightarrow\ &x \text{ nil} \end{aligned}$$

□

Proposition 9.0.5 For any f.p. algebra A, we have that:

$$\mathrm{Nil}(A) = \mathrm{Jac}(A)$$

Proof Assume $a : A$. We have that a is nilpotent if and only if:

$$\forall (x : \mathrm{Spec}(A)).\ a(x) \text{ nil}$$

Now by lemma 9.0.4 this is equivalent to:

$$\forall (x : \mathrm{Spec}(A))(y : R).\ 1 - a(x)y \text{ inv}$$

Which by considering b to be the constant map with value y, is equivalent to:

$$\forall (b : A)(x : \mathrm{Spec}(A)).\ 1 - a(x)b(x) \text{ inv}$$

which is the equivalent to:

$$\forall (b : A).\ 1 - ab \text{ inv}$$

□

10 Automorphisms of projective space

The following should be one part of showing that autormorphisms of $\mathbb{P}^n$ are given by $\mathrm{PGL}_{n+1}(R)$.

Theorem 10.0.1
Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be an arbitrary map. Suppose f is not not in $\mathrm{PGL}_{n+1}(R)$. Then f is in $\mathrm{PGL}_{n+1}(R)$.

Proof First note that f sends any $n+1$ points in general position to $n+1$ points in general position. This is because to be in general position means that a determinant is invertible, which is a negative property, and any map in PGL preserves the property of being in general position.

Let e_i be the point of $\mathbb{P}^n$ with zeroes in all coordinates except the ith. Since $e_0, \cdots, e_n$ are in general position, so are $f(e_0), \cdots, f(e_n)$. Thus we can find $f' \in \mathrm{PGL}$ with $f'(e_i) = f(e_i)$. Replacing f by $f'^{-1} \circ f$, we may assume that $f(e_i) = e_i$. Now if f is in PGL, it must be given by a diagonal matrix, so f is not not given by a diagonal matrix.

Write $U_i \subseteq \mathbb{P}^n$ for the standard affine patch of $\mathbb{P}^n$ consisting of points whose ith coordinate is invertible. We have that $f(U_i) \subseteq U_i$, since to be in U_i is a negative property and this containment holds if f is given by a diagonal matrix. Now f restricts to a map $U_i \to U_i$ which by SQC is given by $n+1$ polynomials in n variables. Homogenising these polynomials, we see that for $x = [X_0 : \cdots : X_n] \in U_i$, $f(x)$ is given by polynomials $p_{i0}, \cdots, p_{in} \in R[X_0, \cdots, X_n]$ homogeneous of some degree d_i, so that $f(x) = [p_{i0} : \cdots : p_{in}]$, where $p_{ii} = X_i^{d_i}$.

Since $f(e_i) = e_i$, we have that the coefficient of $X_i^{d_i}$ in p_{ij} is zero for $i \neq j$. We also have that the coefficient of $X_i^{d_i - 1} X_j$ in p_{ij} is invertible, since this holds when f is given by a diagonal matrix (in this case the coefficient is the ratio of diagonal entries). We also know that $p_{ij} p_{kl} = p_{il} p_{kj}$ for all i, j, k, l since the descriptions of f on all the patches match up.

We claim that p_{ij} is a unit multiple of $X_i^{d_i - 1} X_j$. To this end, we claim that p_{ij} is a sum of monomials which contain neither X_j^2 nor X_k for $k \neq i$, $k \neq j$. We prove both of these claims separately but using the same idea. The idea is that of fixing a monomial ordering, and using the fact that if g, h are monomials with g 'pseudomonic' in the sense that g has invertible leading coefficient, then any bound on the degree of gh gives a bound on the degree of h. We may assume $i \neq j$ in either case.

1. X_j^2: consider the equation $p_{ij}p_{ji} = X_i^{d_i}X_j^{d_j}$. Consider some monomial ordering which is lexicographic first on the degree of X_j and then on X_i. Here p_{ji} is pseudomonic of degree $X_j^{d_j-1}X_i$. Thus p_{ij} has degree at most $X_jX_i^{d_i-1}$ (and indeed that coefficient is assumed invertible). So X_j^2 cannot appear in p_{ij}.

2. X_k where $k \neq i$, $k \neq j$: consider the equation $p_{ij}p_{jk} = p_{ik}X_j^{d_j}$. Consider some monomial ordering which is lexicographic first on the degree of X_k and then on X_j. By the above, p_{jk} is pseudomonic of degree $X_j^{d_j-1}X_k$, and the degree of p_{ik} is at most $X_kX_j^{d_i-1}$. Thus the degree of p_{ij} is at most $X_kX_j^{d_i-1}X_j^{d_j}/(X_j^{d_j-1}X_k) = X_j^{d_i}$. Thus X_k cannot appear.

Given that p_{ij} is a unit multiple of $X_i^{d_i-1}X_j$, it is direct that f is given by a diagonal matrix. This finishes the proof. □

11 Projective module and vector bundle

In this section we prove the that for any f.p. algebra A we have an equivalence between vector bundles over $\mathrm{Spec}(A)$ and f.g. projective A-module.

Lemma 11.0.1 Assume given a f.p. algebra A and B a flat A-algebra, with A-modules M, N such that M is finitely presented. Then we have that:

$$\mathrm{Hom}_A(M, N) \otimes_A B = \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$$

Proof Assume we have an exact sequence:

$$A^m \to A^n \to M \to 0$$

so by applying $\mathrm{Hom}_A(\_, N)$ we get an exact sequence:

$$0 \to \mathrm{Hom}_A(M, N) \to N^n \to N^m$$

Then by applying $\_ \otimes_A B$ to the first sequence we have an exact sequence:

$$B^m \to B^n \to M \otimes_A B \to 0$$

and then by applying $\mathrm{Hom}_B(\_, N \otimes_A B)$ we get an exact sequence:

$$0 \to \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B) \to (N \otimes_A B)^n \to (N \otimes_A B)^m$$

Finally using B flat and applying $\_ \otimes_A B$ to the second sequence, we get an exact sequence:

$$0 \to \mathrm{Hom}_A(M, N) \otimes_A B \to N^n \otimes_A B \to N^m \otimes_A B$$

From this we can conclude. □

Proposition 11.0.2 Let A be a f.p. algebra, then the equivalence:

$$\{\text{bundles of f.p. modules over}\,\mathrm{Spec}(A)\} \simeq \{\text{f.p. } A-\text{modules}\}$$

restricts to an equivalence:

$$\{\text{vector bundles over}\,\mathrm{Spec}(A)\} \simeq \{\text{f.p. projective } A-\text{modules}\}$$

Moreover any finitely generated projective A-module is in fact finitely presented.

Proof Let M be a finitely generated projective A-module. For any $x : \mathrm{Spec}(A)$, we write:

$$M_x := R \otimes_A M$$

Then we can check that for any $x : \mathrm{Spec}(A)$ we have that M_x is projective and finitely generated, therefore M_x is finite free since R is local. So $x \mapsto M_x$ indeed gives a vector bundle.

Conversely let $x \mapsto M_x$ be a vector bundle over $\mathrm{Spec}(A)$, let us write:

$$M := \prod_{x:\mathrm{Spec}(A)} M_x$$

We already know that M is finitely presented.

Since the M_x are free, we know that the exists a finite cover of $\mathrm{Spec}(A)$ by $D(f_i)$ such that for all i we have that M_{f_i} is a free A_{f_i}-module.

Let us prove that the map of A-module:

$$\mathrm{Hom}_A(M, A^n) \to \mathrm{Hom}_A(M, M)$$

is surjective. To do this it is enough to prove that the induced map:

$$(\mathrm{Hom}_A(M, A^n))_{f_i} \to (\mathrm{Hom}_A(M, M))_{f_i}$$

is surjective for all i. But by lemma 11.0.1 we have that this map is isomorphic to:

$$\mathrm{Hom}_{A_{f_i}}(M_{f_i}, A^n_{f_i}) \to \mathrm{Hom}_{A_{f_i}}(M_{f_i}, M_{f_i})$$

Since M_{f_i} is a free A_{f_i}-module, we know that the surjection:

$$A^n_{f_i} \to M_{f_i}$$

is split, therefore the considered map is indeed surjective.

From the fact that the map:

$$\mathrm{Hom}_A(M, A^n) \to \mathrm{Hom}_A(M, M)$$

is surjective we conclude that M is a direct summand of A^n, and therefore it is indeed projective. □

12 Overtness

A type X is *overt* iff X-indexed sums preserve openness, that is iff for every open $U \subseteq X$ the proposition "U is inhabited" is open again. The following proposition emerged at the 2024 Dagstuhl meeting, prompted by and jointly with Andrej Bauer and Martín Escardó:

Proposition 12.0.1 The following statements are equivalent.

1. The ring R is overt.
2. For every polynomial $f : R[X]$, the proposition that f has an anti-zero (a number x such that $f(x) \neq 0$) is open.
3. The ring R is infinite in the sense that for every natural number n, there are n pairwise distinct elements of R.
4. The ring R is infinite in the sense that for every finite list $x_1, \ldots, x_n$ of elements of R, there is an element y distinct from all of the x_i.

Proof Statement 2 is just the special instance of Statement 1 for the case $U = D(f)$. Conversely, Statement 1 follows from Statement 2 because an arbitrary open of R is of the form $\bigcup_{i=1}^n D(f_i)$ and because finite disjunctions of open propositions are open. Trivially, Statement 4 implies Statement 3.

To verify that Statement 3 implies Statement 2, let $f : R[X]$ be a polynomial. By definition, there is an upper bound n of the formal degree of f. By assumption, there are $n+1$ pairwise distinct numbers $r_0, \ldots, r_n$. Then the statement that f has an antizero is equivalent to the finite disjunction $\bigvee_{i=0}^n (f(r_i) \neq 0)$: The "if" direction is trivial, and for the "only if" direction, assume to the contrary that $f(r_0) = \ldots = f(r_n) = 0$. Then f can be factored as $f(X) = (X - r_1) \cdots (X - r_n) \cdot c$. Because $f(r_0) = 0$, we have $c = 0$ and hence $f = 0$. This is a contradiction to f admitting an antizero.

To verify that Statement 2 implies Statement 4, let numbers $x_1, \ldots, x_n$ of R be given. Up to double negation, the monic polynomial $(X - x_1) \cdots (X - x_n) + 1$ has a zero. Hence up to double negation, the polynomial $f(X) = (X - x_1) \cdots (X - x_n)$ has an antizero. By assumption, this statement is open and therefore double negation stable, hence f actually has an antizero. □

Remark 12.0.2 The equivalent conditions of Proposition 12.0.1 are satisfied in case the external base ring k contains, for every natural number n, elements $x_1, \ldots, x_n$ whose pairwise differences are invertible.

Proposition 12.0.3 If R is overt, then every open neighborhood of 0 in R is infinite in the sense that for every finite list $x_1, \ldots, x_n$ of elements, there is an element y distinct from all the x_i.

Proof Let $U \subseteq R$ be an open neighborhood of 0. Then there is a polynomial $f : R[X]$ such that $0 \in D(f) \subseteq U$. Let $x_1, \ldots, x_n$ be elements of U. Up to double negation, the polynomial $(X - x_1) \cdots (X - x_n) \cdot X \cdot f + 1$ has a zero. Such a zero is an element y which is distinct from all the x_i (and from 0). So up to double negation, the polynomial $(X - x_1) \cdots (X - x_n) \cdot X \cdot f$ has an antizero. Because R is overt, the existence of an antizero is (open and hence) double negation stable so that we can conclude that there actually is an antizero. $\square$

Index

References

[CCH23] Felix Cherubini, Thierry Coquand, and Matthias Hutzler. *A Foundation for Synthetic Algebraic Geometry*. 2023. arXiv: 2307.00073 [math.AG]. URL: https://www.felix-cherubini.de/iag.pdf (cit. on p. 6).

[MM12] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer, 2012 (cit. on p. 6).

[Shu10] Michael A. Shulman. “Stack semantics and the comparison of material and structural set theories”. In: *arXiv e-prints*, arXiv:2205.15887 (2010) (cit. on p. 7).