Proper Synthetic Schemes

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Introduction

This is an incomplete draft on work in progress on proper schemes in synthetic algebraic geometry as introduced in [CCH23]. We assume the axioms presented in [CCH23] throughout these notes.

1 Preliminaries

We will use [CCH23][Lemma 4.2.11]:

Lemma 1.0.1 Let C be a closed proposition and U be an open proposition, then $C \to U$ is equivalent to $\neg C \vee U$.

We will also need:

Lemma 1.0.2 Let C be a closed proposition and $U : C \to \text{Open}$ a family of open propositions. Then there merely is an open proposition $\tilde{U} : \text{Prop}$ such that for any $c : C$, $U(c) = \tilde{U}$.

Proof By [CCH23, Theorem 4.2.6], U can be represented by a list $f_1, \ldots, f_n$ of elements of R^C. Since C is a closed proposition, the map $R \to R^C$ is a quotient map, hence surjective. So we can merely find preimages $\tilde{f}_1, \ldots, \tilde{f}_n : R$. Then we take $\tilde{U}$ to be $D(\tilde{f}_1, \ldots, \tilde{f}_n)$. □

2 Compact Types

The following is expected to be analogous to completeness in algebraic geometry. Since it coincides with the defintion of compactness in synthetic topology ([**MISSING**]) due to Martín Escardó, we just call it compact:

Definition 2.0.1 A type X is *compact*, if for any open proposition $U : X \to \text{Open}$ on X, the type $(x : X) \to U(x)$ is open.

Example 2.0.2 All finite types are compact, since a conjuction of open propositions is open.

Lemma 2.0.3 If A is compact and for each $x : A$, $B(x)$ is a compact type, then the dependent sum $(x : A) \times B(x)$ is compact.

Proof Let $U : ((x : A) \times B(x)) \to \text{Open}$ be open. We have to show $(y : (x : A) \times B(x)) \to U(y)$ is open. By currying, this is $(x : A) \to (z : B(x)) \to U(x, z)$. By compactness of each $B(x)$, the type $V_x :\equiv (z : B(x) \to U((x, z))$ is open for all $x : A$. So we have to show $(x : A) \to V_x$ is open, but this is the case by compactness of A. □

Lemma 2.0.4 Any closed proposition is compact.

Proof Let $U : C \to \text{Open}$. Then, by lemma 1.0.2, there merely is $\tilde{U} : \text{Open}$, such that $U(c) = \tilde{U}$ for any $c : C$. Thus $(x : C) \to U(x)$ is equivalent to $C \to \tilde{U}$. By lemma 1.0.1, this is equivalent to $\neg C \vee \tilde{U}$, which is open. □

With this we can generalize lemma 1.0.2 to types:

Proposition 2.0.5 Let X be a type, $C \subseteq X$ a closed subtype and $U \subseteq C$ open. Then there is an open $\tilde{U} \subseteq X$ such that $\tilde{U} \cap C = U$.

Proof We take $\tilde{U}(x) :\equiv (C(x) \to U(x))$. □

Lemma 2.0.6 A closed subtype of a compact type is compact.

Proof By lemma 2.0.4 and lemma 2.0.3. □

Lemma 2.0.7 If A is compact and $f : A \to B$ is such that for all $y : B$, there not not exists $x : A$ with $f(x) = y$, then B is compact.

Proof Let $U : B \to \text{Open}$. The assumption on f can be written as $(y : B) \to \neg\neg\|(x : A) \times f(x) = y\|$. Then, by $\neg\neg$-stability of opens and our assumption:

$$\begin{aligned}(y : B) \to U(x) &= (y : B) \to \neg\neg\|U(y)\| \\ &= (x : A) \to \neg\neg\|U(f(x))\| \\ &= (x : A) \to U(f(x))\end{aligned}$$

Where the latter is open by the assumption that A is compact. □

Corollary 2.0.8 Let $f : A \to B$, then the image of any compact subtype of A is compact.

Proposition 2.0.9 $\mathbb{A}^1$ is not compact.

Proof Assume $\mathbb{A}^1$ is compact. For $a : R$, $x : \mathbb{A}^1$, let $U_a(x) :\equiv (1 + ax \neq 0)$. Note that $(x : \mathbb{A}^1) \to U_a(x)$ is equivalent to $\neg(a \text{ invertible}) = \neg\neg(a = 0)$. So by assumed compactness of $\mathbb{A}^1$, the proposition $\neg\neg(a = 0)$ is open, and therefore the formal disk $\mathbb{D} :\equiv \{x : R \mid \neg\neg(x = 0)\}$ is open.

To see this, assume $\mathbb{D}$ is open and note $\mathbb{D} \cap (\mathbb{A}^1 \setminus \{0\}) = \emptyset$. $\mathbb{A}^1 \setminus \{0\} \subseteq \mathbb{A}^1$ is dense, so $\mathbb{D} = \emptyset$, which contradicts $0 \in \mathbb{D}$. □

So also any type Y that admits a surjection to $\mathbb{A}^1$ is not compact, which includes all $\mathbb{A}^n$ for $n > 0$.

Lemma 2.0.10 A type X is compact if and only if $\|X\|_0$ is compact.

Proof Clear as open in X are equivalent to open in $\|X\|_0$. □

2.1 Compact propositions

Written by Hugo.

Lemma 2.1.1 An open proposition is compact if and only if it is decidable.

Proof If it is decidable then it is compact. Conversely let U be a compact open proposition. Then there merely exists C closed such that $U = \neg C$. Since U is compact and $\bot$ is open, we have that $\neg U$ is open, i.e. $\neg\neg C$ is open. Then:

$$C \to \neg\neg C$$

gives:

$$\neg C \vee \neg\neg C$$

and this is precisely:

$$U \vee \neg U$$

□

Lemma 2.1.2 Let U be an open proposition, and $C : U \to \mathrm{Prop}$ a family of closed propositions. Then $\Sigma_{x:U} C(x)$ is compact if and only if it is closed.

Proof If it is closed then it is compact. Conversely assume $\Sigma_{x:U} C(x)$ is compact. Then U is merely of the form $\neg D$ for D a closed proposition and we have that:

$$\neg U \to \neg(\Sigma_{x:U} C(x))$$

so that:

$$D \to \neg(\Sigma_{x:U} C(x))$$

Since $\Sigma_{x:U} C(x)$ is compact we have that $\neg(\Sigma_{x:U} C(x))$ is open, and then we have that:

$$\neg D \vee \neg(\Sigma_{x:U} C(x))$$

Since $\neg D = U$ we can conclude $\Sigma_{x:U} C(x)$ closed in both cases. □

Lemma 2.1.3 Let X be a separated scheme and $C \subseteq X$ compact, then the complement of C is open.

Proof By separatedness, $x \neq y$ is open for all $x, y : X$, so the complement $(x : X) \times ((y : C) \to x \neq y)$ is open. □

3 Projective Schemes

Definition 3.0.1 A scheme X is *projective* if it merely is a closed subtype of $\mathbb{P}^n$ for some $n \in \mathbb{N}$.

The goal of this section is to prove that projective space $\mathbb{P}^n$ is compact, a classical result of elimination theory. We will first deal with the case $n = 1$, using algebraic methods, and then deduce the general case.

The following lemma can be understood as a version of the Euclidean algorithm for univariate polynomials in the abscence of decidable equality.

Proposition 3.0.2 Let $p_1, \ldots, p_n : R[X]$. Then we can find propositions $b_1, \ldots, b_r$, with each b_i of the form $D(u) \wedge v_1 = \ldots = v_k = 0$, such that $\neg\neg(b_1 \vee \ldots \vee b_r)$, and for any i, we either have that $(p_1, \ldots, p_n) = 0$ if b_i holds, or we have a natural d such that if b_i holds, then $(p_1, \ldots, p_n)$ is principal generated by a degree d monic polynomial.

Proof (sketch) We suppose each p_i is represented by a list of coefficients. If one of these lists is empty, we simply throw it away. If there is no p_i left, then there is nothing left to prove: we take $r = 1$ and $b_1 = D(1)$, and note that $(p_1, \ldots, p_n) = 0$. Thus we may suppose $n \geq 1$, and take i such that the formal degree of p_i is the smallest. Let u be the leading coefficient of p_i. We have $\neg\neg(D(u) \vee u = 0)$. In either case, we can make progress: if $D(u)$ and $n = 1$, then $(p_1, \ldots, p_n)$ is (p_i) with p_i monic; if $D(u)$ and $n > 1$, then we can divide the other p_j by p_i, decreasing their formal degrees; and if $u = 0$ then we can reduce the formal degree of p_i. □

Lemma 3.0.3 Let A be an R-algebra, J an ideal of R, and $x, y : A$ elements such that $xy = 1$. Suppose 1 is in both $J[x]$ and $J[y]$. Then there is $m : J$ such that $m = 1$ in A.

Proof Write $1 = \sum_{i=0}^{n} a_i x^i = \sum_{j=0}^{m} b_j y^j$. Multiplying by y^j, we have $y^j = \sum_{i=0}^{n} a_j x^i y^j$. For $0 \leq j \leq m$, we have $x^i y^j \in \{y^m, y^{m-1}, \ldots, y, 1, x, \ldots, x^n\}$, since $xy = 1$. Hence $\langle y^m, \ldots, 1, \ldots, x^n \rangle = J\langle y^m, \ldots, 1, \ldots, x^n \rangle$. By Nakayama, there is $m : J$ such that $m \cdot 1 = 1$ in A. ☐

Lemma 3.0.4 Let $U_1 : \mathbb{P}^1 \setminus \{0\} \to \mathrm{Open}$ and $U_2 : \mathbb{P}^1 \setminus \{\infty\} \to \mathrm{Open}$ be open subsets of the two affine patches of the projective line. Then there merely is an open proposition φ, such that if $(x : \mathbb{P}^1 \setminus \{0\}) \to U_1(x)$ and $(x : \mathbb{P}^1 \setminus \{\infty\}) \to U_2(x)$ both hold, then φ also holds, and if φ holds, then for all $x : \mathbb{P}^1 \setminus \{0, \infty\}$, $U_1(x) \vee U_2(x)$ holds.

Proof By [CCH23, Theorem 4.2.7], we have $U_1 = D(p_1, \ldots, p_n)$ and $U_2 = D(q_1, \ldots, q_m)$, where $p_i \in R[X]$ and $q_i \in R[Y]$. By applying proposition 3.0.2 twice and combining the results, we can find propositions $b_1, \ldots, b_r$ with each b_i of the form $D(u) \wedge v_1 = \ldots = v_k = 0$, so that for each i, both $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_m)$ are principal when b_i holds (in the strong sense: we know what the degree will be even without knowing b_i).

Consider now an i such that if b_i holds, then $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_m)$ are both the unit ideal. Write b_i as $D(u) \wedge v_1 = \ldots = v_k = 0$. Since $(p_1, \ldots, p_n)$ is the unit ideal if b_i holds, we have that $(R[X]/(p_1, \ldots, p_n))^{b_i} = R[X, u^{-1}]/(p_1, \ldots, p_n, v_1, \ldots, v_k)$ is trivial. Thus $1 \in J[X]$ in $R[X, u^{-1}]/(p_1, \ldots, p_n)$ where $J = \langle v_1, \ldots, v_k \rangle$. In $R[x, y, u^{-1}]/(xy - 1, p_1(x), \ldots, p_n(x), q_1(y), \ldots, q_m(y))$, we thus have $1 \in J[x]$ and $1 \in J[y]$. By lemma 3.0.3, we have $m' : J$ such that $m' = 1$ in this ring. This means we have $m : J$, $N : \mathbb{N}$ such that $u^N = m$ in $R[x, y]/(xy - 1, p_1(x), \ldots, q_m(y))$. We take φ to be the disjunction of $D(u^N - m)$ over all such i.

We first verify that U_1 and U_2 cover their respective affine patches, then φ holds. That is, we have to derive a contradiction from the assumption that $u^N - m = 0$ in R for each i as above. Since our goal is $\neg\neg$-stable, we may suppose given i such that b_i holds. Since U_1 and U_2 cover their respective affine patches, we actually do have that $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_m)$ are both the unit ideal. Thus i must be of the form above. But now u is invertible and $m = 0$, contradicting $u^N - m = 0$.

Next suppose φ holds and $z : \mathbb{P}^1 \setminus \{0, \infty\}$. We have to show $U_1(z) \vee U_2(z)$. We can write $z = [1 : x] = [y : 1]$ with $xy = 1$, so that we have to show $\neg(p_1(x) = \ldots = p_n(x) = q_1(y) = \ldots = q_m(y) = 0)$. Since we assume φ holds, we can assume given an i such that $u^N - m$ is invertible in R. Since $u^N = m$ in the ring $R[x, y](xy - 1, p_1, \ldots, q_m)$, this ring is trivial, so its spectrum is empty, as needed. ☐

Theorem 3.0.5
The projective line $\mathbb{P}^1$ is compact.

Proof Let $U \subseteq \mathbb{P}^1$ be open. Letting $U_1 = U \setminus \{0\}$ and $U_2 = U \setminus \{\infty\}$, take φ as in lemma 3.0.4. We claim that $(x : \mathbb{P}^1) \to U(x)$ is equivalent to $U(0) \wedge U(\infty) \wedge \varphi$, which is clearly open. The forward implication is clear. For the reverse implication, observe that given $x : \mathbb{P}^1$, we have $\neg\neg(x = 0 \vee x = \infty \vee \neg(x = 0 \vee x = \infty))$, and that $U(x)$ is $\neg\neg$-stable. ☐

Theorem 3.0.6
For each n, $\mathbb{P}^n$ is compact.

Proof We have to show that for any R-module V which is free of finite rank, $\mathbb{P}V$ is compact. We induct on $m = \dim V$. For $m \leq 1$, this is clear, since finite types are compact. Thus suppose $m \geq 2$, so that we may pick $u, v : V$ linearly independent.

Say a *flag* in V of rank r is a sequence $0 = W_0 \subseteq W_1 \subseteq \ldots \subseteq W_r$ of submodules of V such that W_{i+1}/W_i is free of rank 1 for each i. Let F_r be the type of flags in V of rank r such that W_1 is contained in $\langle u, v \rangle$.

We claim that F_r is compact for each r. We prove this by induction on r. For $r = 1$, we have $F_1 = \mathbb{P}\langle u, v \rangle = \mathbb{P}^1$, which is compact by theorem 3.0.5. An element of F_{r+1} is given by an element $W_0, \ldots, W_r$ of F_r together with a point of $\mathbb{P}(V/W_r)$. Since V/W_r is free of rank $m - r < m$, $\mathbb{P}(V/W_r)$ is compact by inductive hypothesis. Thus F_{r+1} is compact, since compact types are closed under Σ.

We define a map $F_{m-1} \to \mathbb{P}V^\star$, where $V^\star$ is the dual module of V. A point of $\mathbb{P}V^\star$ is equivalently a surjection $V \to Q$ where Q is free of rank 1, and indeed we have such a surjection $V \to V/W_{m-1}$ for any point of F_{m-1}.

We claim that this map satisfies the assumptions for lemma 2.0.7. Say given a point of $\mathbb{P}V^\star$. We can represent it by a linear map $c : V \to R$. It is not not the case that $c(w) = 0$ for some non-zero $w : \langle u, v \rangle$, for if $c(u) \neq 0$ then we can take $w = v - \frac{c(v)}{c(u)} u$. Given such a w, we can extend it to a basis of $\ker c$, defining the desired point of F_{m-1}.

Putting everything together, we conclude that $\mathbb{P}V^\star$ is compact. Since $V^\star = V$, $\mathbb{P}V$ is also compact as needed. ☐

Theorem 3.0.7
Projective schemes are compact.

Proof By Lemma 2.0.3, Lemma 2.0.4 closed subtypes of a compact type are compact, so any closed subtype of $\mathbb{P}^n$ is compact by Theorem 3.0.6. $\square$

4 Quasi-projective Schemes

Written by Hugo.

Definition 4.0.1 A scheme X is quasi-projective if it merely is a closed in an open in a projective space.

We think we could equivalently define it as an intersection of an open and a closed. Note that quasi-projective schemes are separated. Are all separated schemes quasi-projective?

Lemma 4.0.2 Projective schemes and affine schemes are quasi-projective.

Lemma 4.0.3 A closed or open subscheme of a quasi-projective schemes is itself quasi-projective.

Proof Remember open in closed is intersection of closed and open. $\square$

Proposition 4.0.4 A quasi-projective scheme is compact if and only if it is projective.

Proof Consider a quasi-projective compact scheme X, then we merely have an embedding:

$$i : X \subset \mathbb{P}^n$$

which fibers are of the form $\Sigma_{x:U} C(x)$ with U open and $C(x)$ closed for all x. Since i goes from a compact type to a type with compact identity types (as closed proposition are compact), its fibers are compact (since compact types are closed by dependent sums). Then by lemma 2.1.2 we conclude that i is actually a closed embedding, and X is projective. $\square$

Do we have that any proper (i.e. compact and separated) scheme is projective? Without separatedness this fails as the suspension of an open proposition is a compact compact scheme, but it is not always separated (so not always projective).

5 Projective and quasi-projective schemes are stable under dependent sums

BEWARE, SECTION IN PROGRESS, By Hugo and Felix.

Here we go through the surprisingly long road giving quasi-projective schemes stable under dependent sums. At the moment, for projective schemes to be stable by dependent sums we are missing the following:

- $H^2(\mathbb{P}^n, \mathbb{A}^\times) = 0$
- $\mathrm{Aut}(\mathbb{P}^n) = PGL_{n+1}$
- Any locally free bundle on $\mathbb{P}^n$ is becomes generated by global sections when twisted enough.

5.1 Families of projective spaces being a projectivisation of a bundle

In this section we prove that for any type X, if $H^2(X, \mathbb{A}^\times) = 0$ then any family of projective spaces over X is the projectivisation of a bundle of finite free modules over X. Next two lemmas are about HoTT only, not SAG.

Lemma 5.1.1 Assume given a central extension:

$$0 \to A \to K \to G \to 0$$

Then the fibers of the induced map:

$$f : BK \to BG$$

are BA-torsors.

Proof We know that the fibers are merely inhabited, 0-connected and 1-truncated, because this holds for the fiber over $* : BG$. So we just need to check that for all $x : BG$ and any point in $y : \mathrm{fib}_f(x)$ we have that:

$$(\mathrm{fib}_f(x), y) = (BA, *)$$

Since the fibers are 0-connected and 1-truncated, it is enough to show that:

$$(y =_{\mathrm{fib}_f(x)} y) \simeq_{\mathrm{Group}} A$$

to conclude. This is equivalent to saying that for all $y : BK$, we have:

$$\left(\sum_{p : y =_{BK} y} f(p) = \mathrm{refl}_{f(y)} \right) \simeq_{\mathrm{Group}} A$$

For $y = *$ this is immediate. Then we need to check that the induced conjugation action of K on A is the identity. But this follows from the assumed centrality of A in K. □

Remark 5.1.2 With a bit more work, one can show that central extensions of G by A are classified by pointed maps from BG to B^2A. Traditionally one only state that they have the same 0-truncations, i.e. central extensions up to iso correspond to $H^2(G, A)$.

Lemma 5.1.3 Assume given a central extension:

$$0 \to A \to K \to G \to 0$$

and a type X such that $H^2(X, A) = 0$. Then any map from X to BG factors through BK

Proof By the previous lemma the map $BK \to BG$ is actually of the form:

$$\left(\sum_{x : BG} P(x) \right) \to BG$$

With $P(x)$ a BA-torsor for all $x : BG$.

Given a map $f : X \to BG$, factoring it through f means proving:

$$\prod_{x : X} P(f(x))$$

which in turns means proving that the BA-torsor:

$$P \circ f : X \to B^2A$$

is trivial. But since $H^2(X, A) = 0$, this we merely have such a factorisation for any torsor. □

Proposition 5.1.4 We have a central extension:

$$0 \to \mathbb{A}^\times \to GL_{n+1} \to Aut(\mathbb{P}^n) \to 0$$

Proof TODO, difficult, says $Aut(\mathbb{P}^n) = PGL_{n+1}$. David has a sketch for $n = 1$. □

A projective space is a type merely equal to $\mathbb{P}^k$ for some k.

Corollary 5.1.5 Assume given X such that $H^2(X, \mathbb{A}^\times) = 0$. Then any family of projective spaces over X is the projectivisation of a bundle of finite free modules over X.

Proof There is a well defined function from projective spaces to natural number giving the dimension, as if $\mathbb{P}^m = \mathbb{P}^n$ then we have $m = n$, for example by considering the tangent space at any chosen point. Given a family of projective space over X, we split X as $\Sigma_{n:\mathbb{N}} X_n$ with the family having dimension n on X_n. So we can assume the family having constant dimension.

Now a family of projective spaces of dimension k over X is a map $X \to BAut(\mathbb{P}^n)$. By proposition 5.1.4 and lemma 5.1.3 we know this map lift through BGL_{n+1}, but the map:

$$BGL_{n+1} \to BAut(\mathbb{P}^n)$$

though which we lift is precisely projectivisation, and we can conclude. □

5.2 Projectivisation of a finite free line bundles over a projective space is projective scheme

Lemma 5.2.1 A locally finite free bundle M on a type X is said to be globally generated if there exists $k : \mathbb{N}$ and:

$$\phi : (x : X) \to \mathrm{Hom}_R(R^k, V_x)$$

for all $x : X$ we have ϕ_x surjective.

Lemma 5.2.2 Assume given globally generated bundle V of finite free module of $\mathbb{P}^n$. Then:

$$\sum_{x:\mathbb{P}^n} \mathbb{P}(V_x^\star)$$

is a projective scheme.

Proof Since for all $x : \mathbb{P}^n$ have a surjective morphism:

$$\phi_x : \mathrm{Hom}_R(R^k, V_x)$$

this means we have an injective morphism:

$$\phi_x^\star : \mathrm{Hom}_R(V_x^\star, R^k)$$

Then we have that $\mathbb{P}(V_x^\star)$ is a closed subscheme of $\mathbb{P}^k$, as $V_x^\star$ is finite free. From this we know that:

$$\sum_{x:\mathbb{P}^n} \mathbb{P}(V_x^\star)$$

is a closed subscheme of:

$$\mathbb{P}^n \times \mathbb{P}^k$$

which is a projective scheme by the Segre embedding **??**. □

Lemma 5.2.3 Assume V is a finite free module. Then for all finite free module L of dimension 1 we have that:

$$\mathbb{P}(V) = \mathbb{P}(\mathrm{Hom}_R(L, V))$$

Proof We define a map:

$$\phi : \mathbb{P}(\mathrm{Hom}_R(L, V)) \to \mathbb{P}(V)$$

by sending any non-zero map $v : \mathrm{Hom}_R(L, V)$ to the image of v in V. It is straightforward to check that the image is a line, and that for $\lambda : R^\times$ we have that λv and v have the same image, so this map is well-defined. When checking that this map is an equivalence we can assume that $L = R$. Then ϕ is the projectivisation of the isomorphism:

$$\mathrm{Hom}_R(R, V) \simeq V$$

so it is an equivalence. □

Lemma 5.2.4 Assume given of locally finite free bundle on $\mathbb{P}^n$. Then for d large enough we have that $V(d)$ is globally generated.

Proof TODO, hard... Know how to do it for sums of twisted canonical line bundles. □

Lemma 5.2.5 Assume given a bundle V of finite free module over $\mathbb{P}^n$. Then:

$$\sum_{x:\mathbb{P}^n} \mathbb{P}(V_x)$$

is a projective scheme.

Proof By lemma 5.2.4 we know that $V^\star(d)$ is globally generated for d large enough. Then by lemma 5.2.2 we know that:

$$\sum_{x:\mathbb{P}^n} \mathbb{P}((V^\star(d))_x^\star)$$

is a projective scheme. Finally, for all $x : \mathbb{P}^n$ we have that:

$$\mathbb{P}((V^\star(d))_x^\star) = \mathbb{P}(\mathrm{Hom}_R(\mathcal{O}(d)_x, V_x)) = \mathbb{P}(V_x)$$

by lemma 5.2.3 because $\mathcal{O}(d)_x$ is free of dimension 1. So we can conclude. □

5.3 Projective schemes are stable under dependent sum

Lemma 5.3.1 We have that:

$$H^2(\mathbb{P}^n, \mathbb{A}^\times) = 0$$

Proof TODO ☐

Proposition 5.3.2 Assume given X a projective scheme and Y_x a family of projective schemes for $x : X$. Then the scheme $\sum_{x:X} Y_x$ is projective.

Proof If X is a closed proposition this is clear because closed propositions have choice. Then it is enough to show that for any family of projective spaces over $\mathbb{P}^n$ their total space is a projective scheme. This holds since by lemma 5.3.1 and corollary 5.1.5 the family is a projectivisation of a bundle of finite free modules, and then we conclude by lemma 5.2.5. ☐

5.4 Projectivisation of a finite free line bundle over an open proposition is a quasi-projective scheme

TODO

5.5 Quasi-projective schemes are stable under dependent sum

TODO

6 Constructible Sets

In this section we introduce constructible sets and prove a version of Chevalley's theorem.

Lemma 6.0.1 For a proposition P, the following are equivalent:

- P can be expressed as $\neg\neg(b_1 \vee \ldots \vee b_n)$, with each b_i the conjunction of a standard open proposition and a closed proposition
- P can be expressed as $\neg c_1 \wedge \ldots \wedge \neg c_m$, with each c_j the conjunction of a standard open proposition and closed proposition.

We say P is *constructible* if either condition merely holds. For a type X, we say a subtype $C : X \to \mathrm{Prop}$ is constructible if the proposition $C(x)$ is constructible for each $x : X$.

Proof Note that the second form is equivalent to $\neg(c_1 \vee \ldots \vee c_m)$. An expression like $b_1 \vee \ldots \vee b_n$ can be understood as the disjunctive normal form of a propositional formula built from propositions of the form $v = 0$. Hence, assuming excluded middle, the negation of such an expression is again of the same form. Since we are dealing only with negated statement, we may assume excluded middle, giving the desired result. ☐

Lemma 6.0.2 We have the following.

- Open propositions are constructible.
- The type of constructible propositions is closed under negation and conjunction.
- Constructible propositions are $\neg\neg$-stable.
- The type of constructible propositions is a Boolean algebra, with join given by $\neg\neg(\varphi \vee \psi)$.
- This is a Boolean subalgebra of the Boolean algebra of $\neg\neg$-stable propositions.

The proof is direct in each case. Constructible propositions are not closed under disjunction, for if $\neg\neg(u = 0) \vee \neg\neg(v = 0)$ were constructible, it would be equivalent to $\neg\neg(uv = 0)$, but it is is even stronger than $uv = 0$. We will soon see that the Boolean algebra of constructible propositions enjoys a certain universal property, namely that of Joyal's constructible spectrum. First we need a more general lemma.

Lemma 6.0.3 Let L be a distributive lattice and B a boolean algebra. Let $f : L \to B$ be a map of lattices, and suppose

- each element of B is generated from elements in the image of f using logical connectives
- f is an embedding.

Then f exhibits B as the free Boolean algebra on the distributive lattice L.

Proof We need to show that B does not introduce more relations than necessary on expressions obtained from the image of f. Thus suppose given such a relation $x \leq y$ in B. Writing x in disjunctive normal and y in conjunctive normal form, it suffices to consider a relation of the form $f(a) \wedge \neg f(b) \leq f(c) \vee \neg f(d)$. This is equivalent by Boolean algebra laws to $f(a) \wedge f(d) \leq f(c) \vee f(b)$, that is to $f(a \wedge d) \leq f(c \vee b)$. Since f is an embedding, we have $a \wedge d \leq c \vee b$ already in L, as needed. □

Theorem 6.0.4
For a scheme X, the type of constructible subsets of X is the free Boolean algebra on the lattice of open subsets of X.

Proof We apply lemma 6.0.3. To verify the first condition, that constructible sets are generated by open sets by applying logical connectives, we pick an affine cover of X and on each patch apply an argument analogous to the proof of [CCH23, Theorem 4.2.7]. The second condition, that open subsets embed in constructible subsets, is direct. Note that the inclusion of open subsets into construtible subsets is a lattice map, since $\varphi \vee \psi$ is equivalent to $\neg\neg(\varphi \vee \psi)$ for φ, ψ open propositions. □

Finally we prove the main result about constructible subsets, a version of Chevalley's theorem.

Theorem 6.0.5
Let X be a scheme and $C : X \to \mathrm{Prop}$ a constructible subset of X. Then $(x : X) \to C(x)$ is constructible.

It follows by duality that $\neg\neg(x : X) \times C(x)$ is constructible. This is closer to the usual statement that the image of a constructible set is constructible.

Proof We may assume $X = \operatorname{Spec} A$ is affine, by picking an affine cover and using that constructible propositions are closed under conjunction. Writing $\operatorname{Spec} A$ as a sigma-type and currying, it suffices to consider two cases: $A = R[x]$ and $A = R/(a)$. In each case we apply theorem 6.0.4 to write C in the form $x \mapsto \neg c_1(x) \wedge \ldots \wedge \neg c_n(x)$ where c_1 is the intersection of a standard open subset of $\operatorname{Spec} A$ and a closed subset of $\operatorname{Spec} A$.

In the case of $A = R/(a)$, we use surjectivity of the quotient map $R \to R/(a)$ to find one constructible proposition φ such that $C(x) = \varphi$ for all $x : X$. We claim that $(x : X) \to C(x)$ is equivalent to $\neg\neg(D(a) \vee \varphi)$, which is constructible. To see that these are equivalent, we note that both propositions are $\neg\neg$-stable, so we can prove the equivalence assuming law of excluded middle, which makes it direct.

Now consider the case of $A = R[X]$. Since constructible propositions are closed under conjunction, we may suppose C is of the form $x \mapsto \neg(D(p(x)) \wedge q_1(x) = \ldots = q_n(x) = 0)$, with $p, q_1, \ldots, q_n : R[X]$. Then $(x : X) \to C(x)$ is equivalent to the ring $R[X, p^{-1}]/(q_1, \ldots, q_n)$ being trivial, hence to p being in the radical of $(q_1, \ldots, q_n)$. We apply proposition 3.0.2 to $q_1, \ldots, q_n$. We explain what to do when $(q_1, \ldots, q_n)$ is zero or generated by one monic polynomial q. In the first case, $(x : X) \to C(x)$ is equivalent to the assertion that each coefficient of p is nilpotent, which is a constructible proposition.

In the second case, say q is monic of degree d. We can divide p^d by q to obtain a list of d elements of R. We claim that $(x : X) \to C(x)$ is equivalent to $\neg\neg(q \mid p^d)$, which is constructible since it asserts that some list of d numbers is not not zero. The reverse implication is direct, using that $C(x)$ is $\neg\neg$-stable. For the forward implication, we suppose $(x : X) \to C(x)$, so that $q \mid p^N$ for some N, and want to derive a contradiction from the assumption $\neg(q \mid p^d)$. We appeal to [CCH23, lemma 3.4.3] to factorise q into powers of distinct linear factors. Since q has degree d, each exponent in this factorisation is at most d. Since $q \mid p^N$, each linear factor must not not divide p. Supposing each linear factor divides p, q must divide p^d since the linear factors are coprime, as needed. □

While it is not clear how to characterise open propositions among constructible propositions, we do have the following result.

Lemma 6.0.6 Let P be a proposition of the form $(u_1 \neq 0 \wedge I_1 = 0) \vee \ldots \vee (u_n \neq 0 \wedge I_n = 0)$ where I_i are finitely generated ideals of R. Then P is open if and only if P is equivalent to P' where

$$P' :\equiv (u_1 \neq 0 \wedge I_1^2 = 0) \vee \ldots \vee (u_n \neq 0 \wedge I_n^2 = 0)$$

if and only if P is double negation stable.

Morally, this expresses the idea that an open subset is one which does not get bigger after 'infinitesimal fattening'.

Proof Clearly if P is open then it is double negation stable. We have $P \to P'$ and $P' \to \neg\neg P$, so if P is double negation stable then P is equivalent to P'. It remains to show that if $P' \to P$, then P is open. We prove this by induction on n. We have $P \to (u_1 \neq 0) \vee \ldots \vee (u_n \neq 0)$, where the latter is open. Since open propositions are closed under sigma, we may suppose that $(u_1 \neq 0) \vee \ldots \vee (u_n \neq 0)$ holds. Without loss of generality, say $u_1 \neq 0$ (so that in particular $n \geq 1$). Thus P is equivalent to $(I_1 = 0) \vee Q$, where $Q \coloneqq (u_2 \neq 0 \wedge I_2 = 0) \vee \ldots$, and P' is equivalent to $(I_1^2 = 0) \vee Q'$, with Q' defined in the evident way.

Since $P' \to P$, we have in particular $(I_1^2 = 0) \to (I_1 = 0) \vee Q$. By the following lemma, this means $(I_1^2 = 0) \to (I_1 = 0)$ or $(I_1^2 = 0) \to Q$. In the first case, $I_1 = 0$ is decidable by [Che+23]. If $I_1 = 0$ is true, then P is true, and hence decidable. If $I_1 \neq 0$, then P is equivalent to Q and P' is equivalent to Q', so Q is equivalent to Q' and we are done by inductive hypothesis. In the second case, where $(I_1^2 = 0) \to Q$, we again have that P is equivalent to Q and P' to Q', so we are done for the same reason. □

Lemma 6.0.7 Let P be a closed proposition and A, B arbitrary propositions. Then $P \to (A \vee B)$ is equivalent to $(P \to A) \vee (P \to B)$.

This captures the idea that closed propositions are not disjunctive, i.e. cannot be partitioned.

Proof The reverse implication is direct. So suppose $P \to (A \vee B)$. We claim $P \to (A + B)$. To see this, we apply Zariski local choice together with the fact that R^P has no Zariski covers (it is essentially a local ring, being a quotient of R, except we do not know $0 \neq 1$). This determines a map $P \to \mathrm{Bool}$, and hence by composition a map $P \to R$. Again, since R^P is a quotient of R, we find one element $r : R$ representing our function $P \to \mathrm{Bool}$. Since R is local, either r or $1 - r$ is invertible. In either case we have an element of Bool such that the map $P \to \mathrm{Bool}$ is constantly this element. This means $(P \to A) + (P \to B)$, as needed. □

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