Projective Space in Synthetic Algebraic Geometry

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April 26, 2024

Introduction

The goal of this note is to show that $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$ in the setting of Synthetic Algebraic Geometry [CCH23]. We actually present a strengthening of this result, which in particular states the equivalence

$$(\mathbb{P}^n \to K(R^\times, 1)) \simeq \mathbb{Z} \times K(R^\times, 1)$$

One application is that $\mathrm{Aut}(\mathbb{P}^n)$ is PGL_{n+1}.

For the case $n = 1$, we follow the proof of Horrock's Lemma as presented in Lam's book on Serre's problem [Lam06] on projective modules[1]. For the general case, we don't follow the Quillen patching technique presented in the 1976 paper [Qui76], but instead present an argument which uses our description of $\mathbb{P}^1 \to K(R^\times, 1)$. We then explain how we can deduce that $\mathrm{Aut}(\mathbb{P}^n)$ is PGL_{n+1}.

One point of this work is to show that all these results can be proven axiomatically in the setting of univalent type theory with the 3 axioms described in [CCH23].

1 Definition of $\mathbb{P}^n$ and some linear algebra

We follow the notations and setting for Synthetic Algebraic Geometry [CCH23]. In particular, R denotes the generic local ring and $R^\times$ is the multiplicative group of units of R.

In Synthetic Algebraic Geometry, a scheme is defined as a set satisfying some property [CCH23]. In particular the projective space $\mathbb{P}^n$ can be defined to be the quotient of $R^{n+1} \setminus \{0\}$ by the equivalence relation $a \sim b$ which expresses that a and b are proportional, which is equal to $\Sigma_{r:R^\times} ar = b$. We can then prove [CCH23] that this set is a scheme. This definition goes back to [Koc].

In this setting, a map of schemes is simply an arbitrary set theoretic map. An application of this work is to show that the maps $\mathbb{P}^n \to \mathbb{P}^m$ are given by $m + 1$ homogeneous polynomials of the same degree in $n + 1$ variables.

There is another definition of $\mathbb{P}^n$ which uses "higher" notions. Let $K(R^\times, 1)$ be the delooping of $R^\times$. It can be defined as the type of lines $\Sigma_{M:R\text{-Mod}} \|M = R^1\|$. Over $K(R^\times, 1)$ we have the family of sets

$$T_n(l) = l^{n+1} \setminus \{0\}$$

Note that we use the same notation for an element $l : K(R^\times, 1)$, its underlying R-module and its underlying set. An equivalent definition of $\mathbb{P}^n$ is then

$$\mathbb{P}^n = \sum_{l:K(R^\times,1)} T_n(l)$$

That is, we replaced the quotient, here a set of orbits for a free group action, by a sum type over the delooping of this group [Bez+]. More explicitly, we will use the following identifications:

Remark 1.1 Projective n-space $\mathbb{P}^n$ is given by the following equivalent constructions of which we prefer the first in this article:

(i) $\sum_{l:K(R^\times,1)} T_n(l)$

(ii) The set-quotient $R^{n+1} \setminus \{0\}/R^\times$, where $R^\times$ acts on non-zero vectors in R^{n+1} by multiplication.

[1] This argument is different from the one presented in Lombardi-Quitté [LQ15]; we instead give a constructive version of the proof of Nashier-Nichols [NN87].

(iii) For any k and R-module V we define the *Grassmannian*

$$\mathrm{Gr}(k, V) :\equiv \{U \subseteq V \mid U \text{ is an } R\text{-submodule and } \|U = R^k\| \}.$$

Projective n-space is then $\mathrm{Gr}(1, R^{n+1})$.

We use the following, well-defined identifications:

(i)→(iii): Map (l, s) to $R \cdot (us_0, \ldots, us_n)$ where $u : l = R^1$

(iii)→(i): Map $L \subseteq R^{n+1}$ to (R^1, x) for a non-zero $x \in L$

(ii)↔ (iii): A line through a non-zero $x : R^{n+1}$ is identified with $[x] : R^{n+1} \setminus \{0\}/R^\times$

We construct the standard line bundles $\mathcal{O}(d)$ for all $d \in \mathbb{Z}$, which are classically known as *Serre's twisting sheaves* on $\mathbb{P}^n$ as follows:

Definition 1.2 For $d : \mathbb{Z}$, the line bundle $\mathcal{O}(d) : \mathbb{P}^n \to K(R^\times, 1)$ is given by $\mathcal{O}(d)(l, s) = l^{\otimes d}$ and the following definition of $l^{\otimes d}$ by cases:

(i) $d \geqslant 0$: $l^{\otimes d}$ using the tensor product of R-modules

(ii) $d < 0$: $(l^\vee)^{-d}$, where $l^\vee :\equiv \mathrm{Hom}_{R\text{-Mod}}(l, R^1)$ is the dual of l.

This definition of $\mathcal{O}(d)$ agrees with [CCH23][Definition 6.3.2] where $\mathcal{O}(-1)$ is given on $\mathrm{Gr}(1, R^{n+1})$ by mapping submodules of R^{n+1} to $K(R^\times, 1)$. Using the identification of $\mathbb{P}^n$ from Remark 1.1 we can give the following explicit equality:

Remark 1.3 We have a commutative triangle:

$$\begin{tikzcd} (l : K(R^\times, 1)) \times (T_n(l)) \arrow[rr] \arrow[dr, "\mathcal{O}(1)"'] & & R^{n+1} \setminus \{0\}/R^\times \arrow[dl, "\mathcal{O}(1)"] \\ & K(R^\times, 1) & \end{tikzcd}$$

by the isomorphism given for (l, s) by mapping $x : l$ to $r(us_0, \ldots, us_n) \mapsto r(ux)$ for some isomorphism $u : l \cong R^1$.

Connected to this definition of $\mathbb{P}^n$, we will prove some equalities in the following. To prove these equalities, we will make use of the following lemma, which holds in synthetic algebraic geometry:

Lemma 1.4 Let $n, d : \mathbb{N}$ and $\alpha : R^n \to R$ be a map such that

$$\alpha(\lambda x) = \lambda^d \alpha(x)$$

then α is a homogenous polynomial of degree d.

Proof By duality, any map $\alpha : R^n \to R$ is a polynomial. To see it is homogenous of degree d, let us first note that any $P : R[\lambda]$ with $P(\lambda) = \lambda^d P(1)$ for all $\lambda : R^\times$ also satisfies this equation for all $\lambda : R$ and is therefore homogenous of degree d. Then for $\alpha'_x : R[\lambda]$ given by $\alpha'_x(\lambda) :\equiv \alpha(\lambda \cdot x)$ we have $\alpha'_x(\lambda) = \lambda^d \alpha'_x(1)$. This means any coeffiecent of α'_x of degree different from d is 0. Since this means every monomial appearing in α, which is not of degree d, is zero for all x and therefore 0. □

Proposition 1.5

$$\prod_{l:K(R^\times,1)} l^n \to l \quad = \quad \mathrm{Hom}(R^n, R)$$

Proof We rewrite $\mathrm{Hom}(R^n, R)$, the set of R-module morphism, as

$$\sum_{\alpha:R^n \to R} \prod_{\lambda:R^\times} \prod_{x:R^n} \alpha(\lambda x) = \lambda \alpha(x)$$

using Lemma 1.4 with $d = 1$.

It is then a general fact that if we have a pointed connected groupoid (A, a) and a family of sets $T(x)$ for $x : A$, then $\prod_{x:A} T(x)$ is the set of fixedpoints of $T(a)$ for the $(a = a)$ action [Bez+]. □

We will use the following remark, proved in [CCH23][Remark 6.2.5].

Lemma 1.6 Any map $R^{n+1} \setminus \{0\} \to R$ can be uniquely extended to a map $R^{n+1} \to R$ for $n > 0$.

We will also use the following proposition, already noticed in [CCH23].

Proposition 1.7 Any map from $\mathbb{P}^n$ to R is constant.

Proof Since $\mathbb{P}^n$ is a quotient of $R^{n+1} \setminus \{0\}$, the set $\mathbb{P}^n \to R$ is the set of maps $\alpha : R^{n+1} \setminus \{0\} \to R$ such that $\alpha(\lambda x) = \alpha(x)$ for all λ in $R^\times$. These are exactly the constant maps using Lemma 1.6 and Lemma 1.4 with $d = 0$. $\square$

Proposition 1.8 For all $n : \mathbb{N}$ we have:

$$\prod_{l:K(R^\times,1)} T_n(l) \to T_n(l) \quad = \quad \mathrm{GL}_{n+1}$$

Proof For $n = 0$, this is the direct computation that a Laurent-polynomial $\alpha : (R[X, 1/X])^\times$ which satisfies $\alpha(\lambda x) = \lambda\alpha(x)$ is $\lambda\alpha(1)$ where $\alpha(1) : R^\times = \mathrm{GL}_1$.

For $n > 0$, the proposition follows from two remarks.

The first remark is that maps $T_n(R) \to T_n(R)$, which are invariant under the induced $K(R^\times, 1)$ action, are linear. To prove this remark, we first map from $T_n(l) \to T_n(l)$ to $T_n(l) \to l^{n+1}$ by composing with the inclusion. Maps of the latter kind can be uniquely extended to maps $l^{n+1} \to l^{n+1}$, since by Lemma 1.6 the restriction map

$$(l^{n+1} \to l) \to ((l^{n+1} \setminus \{0\}) \to l)$$

is a bijection for $n > 0$ and all $l : K(R^\times, 1)$.

The second remark is that a linear map $u : R^m \to R^m$ such that

$$x \neq 0 \;\to\; u(x) \neq 0$$

is exactly an element of GL_m.

We show this by induction on m. For $m = 1$ we have $u(1) \neq 0$ iff $u(1)$ invertible.

For $m > 1$, we look at $u(e_1) = \Sigma\alpha_i e_i$ with $e_1, \dots, e_m$ basis of R^m. We have that some α_j is invertible. By composing u with an element in GL_m, we can then assume that $u(e_1) = e_1 + v_1$ and $u(e_i) = v_i$, for $i > 1$, with $v_1, \dots, v_m$ in $Re_2 + \cdots + Re_m$. We can then conclude by induction. $\square$

We can generalize Proposition 1.5 and get a result related to Proposition 1.8 as follows.

Lemma 1.9 (i)

$$\prod_{l:K(R^\times,1)} l^n \to l^{\otimes d} \quad = \quad (R[X_1, \dots, X_n])_d$$

That is, every element of the left-hand side is given by a unique homogeneous polynomial of degree d in n variables.

(ii) An element in

$$\prod_{l:K(R^\times,1)} T_n(l) \to T_m(l^{\otimes d})$$

is given by $m + 1$ homogeneous polynomials $p = (p_0, \dots, p_m)$ of degree d such that $x \neq 0$ implies $p(x) \neq 0$.

Proof We show the first item. Following [Bez+] again, this product is the set of maps $\alpha : R^n \to R^{\otimes d}$ which are invariant by the $R^\times$-action which in this case acts by mapping α to $r^d\alpha(r^{-1}x)$ for each $r : R^\times$. So by Lemma 1.4 these are exactly the maps given by homogeneous polynomials of degree d. $\square$

2 Horrocks Theorem

We will need the following special case of Horrocks Theorem [Lam06; LQ15], for a commutative ring A.

Lemma 2.1 If an ideal of $A[X]$ divides a principal ideal (f) with f monic then it is itself a principal ideal.

Let I and J be such that $I \cdot J = (f)$. We can then write $f = \Sigma u_i v_i$ with u_i in I and v_i in J. We then have $I = (u_1, \dots, u_n)$ and $J = (v_1, \dots, v_n)$. The strategy of the proof is to build comaximal monoids $S_1, \dots, S_l$ in A [LQ15] such that I is generated by a monic polynomial in each $A_{S_j}[X]$.

2.1 Formal computation of gcd

We start by describing a general technique introduced in [LQ15].

If we have a list $u_1, \dots, u_n$ of polynomials over a field we can compute the gcd so that $(g) = (u_1, \dots, u_n)$ and g is 0 or a monic polynomial.

In general if we are now over a ring R, we can interpret this computation formally as follows. We build a binary tree of root R. At each node of the tree we have a f.p. extension A of R. If we want to decide whether an element a in R is invertible or 0 [2] we open two branches: one with $A \to A/(a)$ (intuitively we force a to be 0) and the other with $A \to A_a = A[1/a]$ (intuitively we force a to be invertible).

In this way we have at each leaf a f.p. extension $R \to A$ and in A we have g, a monic polynomial in $A[X]$ or 0, such that $(g) = (u_1, \dots, u_n)$ in $A[X]$. Over each branch we have a list of elements $a_1, \dots, a_n$ of R that we force to be invertible, and a list of elements $b_1, \dots, b_m$ of R that we force to be 0. We associate to this branch the multiplicative monoid generated by $a_1 \dots a_n$ and $1 + (b_1, \dots, b_m)$. In this way, we build a list of monoids $S_1, \dots, S_l$ that are *comaximal* [LQ15]: if s_i in S_i then $1 = (s_1, \dots, s_l)$.

2.2 Application to Horrocks' Theorem

We assume $f = \Sigma u_i v_i$ and $f p_{ij} = u_i v_j$ with $\Sigma p_{ii} = 1$ in $A[X]$. The goal is to build comaximal monoids $S_1, \dots, S_l$ with I generated by a monic polynomial in $A_{S_j}[X]$.

We first build a binary tree which corresponds to the formal computation of the gcd of $u_1, \dots, u_n$ as described above. To each branch we associate an element that we force to be invertible and a list of elements $b_1, \dots, b_m$ that we force to be 0. We write S for the multiplicative monoid generated by a and $1 + (b_1, \dots, b_m)$. We also have a monic polynomial γ in $A_S[X]$ such that $I = (\gamma)$ in $A_S[X]/(b_1, \dots, b_m)$.

Note that $I = (u_1, \dots, u_n)$ contains f.

Lemma 2.2 If p is a polynomial in I which is monic in $A_S[X]/(b_1, \dots, b_m)$ of degree $< deg(f)$ then there exists h monic in $A_S[X]$ and in $(u_1, \dots, u_n)$ and such that $p = h \bmod (b_1, \dots, b_m)$.

Proof (Same proof as in Lam [Lam06].) Let N be the degree of f. If I also contains a polynomial q which is monic mod. L of degree $N - 1$, we can kill all coefficients (in L) of degree $\geqslant N$ using f, and we get that I also contains a monic polynomial of degree $N - 1$ and equal to q mod. L. Similarly I will also contain a monic polynomial of degree $N - 2$, and so on, until we get h monic in $(u_1, \dots, u_n)$ and equal to p mod. L. □

By this Lemma, we get a monic polynomial h in $(u_1, \dots, u_n)$ in $A_S[X]$ and such that $I = (h)$ in $A_S[X]/(b_1, \dots, b_m)$.

Lemma 2.3 $I = (h)$ in $A_S[X]$.

Proof Let L be $(b_1, \dots, b_m)$ in $A_S[X]$. Since I contains $I \cap L$ and $I \cdot J = (f)$ with f regular, we can find K such that $I \cdot K = I \cap L$. We then have $I \cdot K = 0$ mod. L and hence $K = 0$ mod. L since I contains f which is monic. This means $I \cap L = I \cdot L$. Then we have $I = (h) + I \cdot L$. The result then follows from the fact that h is monic and from Nakayama, as in Lam [Lam06]: the module $M = I/(h)$ is a finitely generated module over A_S and satisfies $M \subseteq ML$. □

Corollary 2.4 We can find comaximal elements $s_1, \dots, s_l$ such that I is principal and generated by a monic polynomial in each $A_{s_j}[X]$. Since these monic polynomials are uniquely determined we can patch these generators and get that I is principal in $A[X]$[3].

3 Line bundles on affine schemes

A line bundle on a type X is a map $X \to K(R^\times, 1)$.

A line bundle L on $\mathrm{Spec}(A)$ will define a f.p. A-module $\prod_{x:\mathrm{Spec}(A)} L(x)$ [CCH23]. It is presented by a matrix P. Since this f.p. module is locally free, we can find Q such that $PQP = P$ and $QPQ = Q$ [LQ15]. We then have $Im(P) = Im(PQ)$ and this is a projective module of rank 1. We can then assume P square matrix and $P^2 = P$ and the matrix $I - P$ can be seen as listing the generators of this module.

[2] A priori it is an element of A, but we can always assume that this element comes from an element of R.

[3] If A is not connected, the generator of I may not be monic: if $e(1 - e) = 1$ then the ideal $(eX + (1 - e))$ divides the ideal (X).

If M is a matrix we write $\Delta_l(M)$ for the ideal generated by the $l \times l$ minors of M. We have $\Delta_1(I-P) = 1$ and $\Delta_2(I-P) = 0$, since this projective module is of rank 1.

The module is free exactly if we can find a column vector X and a line vector Y such that $XY = I-P$. We then have $YX = 1$, since if $r = YX$ we have $I - P = XYXY = rXY = r(I-P)$ and hence $r = 1$ since $\Delta_1(I-P) = 1$.

The line bundle on $\mathrm{Spec}(A)$ is trivial on $D(f)$ if, and only if, the module $M \otimes A[1/f][X]$ is free, which is equivalent to the fact that we can find X and Y such that $YX = (f^N)$ and $XY = f^N(I-P)$ for some N.

We can then apply Horrocks Lemma 2.1 in Synthetic Algebraic Geometry for the ring R.

Proposition 3.1 If we have $L : \mathbb{A}^1 \to K(R^\times, 1)$ which is trivial on some $D(f)$ where f in $R[X]$ is monic then L is trivial on $\mathbb{A}^1$.

Corollary 3.2 If we have $L : \mathbb{P}^1 \to K(R^\times, 1)$ then we have

$$\|\prod_{r:R} L([1:r]) = L([1:0])\| \qquad \|\prod_{r:R} L([r:1]) = L([0:1])\|$$

Proof By Zariski local choice [CCH23], the line bundle L is locally trivial. On one chart of $\mathbb{P}^1$, L is trivial on a neighborhood U of 0, so we get $g : R[X]$ such that $g(0) \neq 0$ and L is trivial on $D(g)$. Passing to the other chart, there is some N such that $f :\equiv f(0)^{-1} \cdot g(1/X) \cdot X^N$ is a monic polynomial and L is trivial on $D(f)$, since $D(f) \subseteq U$. $\square$

4 Picard group of $\mathbb{P}^1$

The following result also holds for a general *connected*[4] ring, without assuming a finite presentation.

Lemma 4.1 Let A be a connected, finitely presented R-algebra, then an invertible element of $A[X, 1/X]$ can be written $X^N \Sigma a_n X^n$ with N in $\mathbb{Z}$ and a_0 unit and a_n nilpotent if $n \neq 0$.

Proof Let $P : A[X, 1/X]$ be invertible, with inverse $Q : A[X, 1/X]$, and $P = \sum_i a_i X^i$ and $Q = \sum_i b_i X^i$. By duality we can view the coefficients as functions $a_i, b_i : \mathrm{Spec}(A) \to R$. For all $x : \mathrm{Spec}(A)$, we get an invertible $P_x : R[X, 1/X]$ by evaluating the coefficients of P at x. Then $P_x \cdot Q_x = 1$ and in particular $1 = \sum_{i+j=0} a_i b_j$, so by locality of R, we have i such that a_i is invertible. Without loss of generality, we assume $i = 0$ and want to show $\neg\neg(a_j = 0)$ for $j \neq 0$. Since we prove a negated proposition, we can assume that we have l, k minimal with a_l and b_k invertible. Then we must have $k + l = 0$ because we would have $0 = a_l b_k$ otherwise. k was minimal, so it is 0 and l is 0 as well. The same reasoning applies for a maximal choice of k. $\square$

Using this Lemma we deduce the following.

Lemma 4.2 Any invertible element of $A[X, 1/X]$ can be written uniquely as a product $uX^l(1+a)(1+b)$ with l in $\mathbb{Z}$, u in $A^\times$ and a (resp. b) polynomial in $A[X]$ (resp. $1/XA[1/X]$) with only nilpotent coefficients.

Proof Write $\Sigma v_n X^n$ the invertible element of $A[X, 1/X]$. W.l.o.g. we can assume that the polynomial is of the form $1 + \Sigma v_n X^n$ with all v_n, $n \in \mathbb{Z}$ nilpotent. We let J be the ideal generated by these nilpotent elements. We have some N such that $J^N = 0$.

We first multiply by the inverse of $1 + \Sigma_{n<0} v_n X^n$, making all coefficients of X^n, $n < 0$ in J^2. We keep doing this until all these elements are 0. We have then written the invertible polynomials on the form $(1+a)(1+b)$.

Such a decomposition is unique: if we have $(1+a)(1+b)$ in $A^\times$ with $a = \Sigma_{n \geqslant 0} a_n X^n$ and $b = \Sigma_{n<0} b_n X^n$ then we have $a_n = 0$ for $n > 0$ and $b_n = 0$ for $n < 0$. $\square$

Corollary 4.3 We have $\prod_{L:\mathbb{P}^1 \to K(R^\times,1)} \Sigma_{p:\mathbb{Z}} \|L = \mathcal{O}(p)\|$

Proof A line bundle $L([x_0, x_1])$ on $\mathbb{P}^1$ is trivial on each of the affine charts $x_0 \neq 0$ and $x_1 \neq 0$ by Corollary 3.2, so it is characterised by an invertible Laurent polynomial on R, and the result follows from Lemma 4.2. $\square$

[4] If $e(1-e) = 0$ then $e = 0$ or $e = 1$.

We can then state the following strengthening.

Proposition 4.4 The map $K(R^\times,1)\times\mathbb{Z}\to(\mathbb{P}^1\to K(R^\times,1))$ which associates to (l_0,d) the map $x\mapsto l_0\otimes\mathcal{O}(d)(x)$ is an equivalence.

Proof Corollary 4.3 shows that this map is surjective. So we can conclude by showing that the map is also an embedding. For $(l,d),(l',d'):K(R^\times,1)\times\mathbb{Z}$ let us first consider the case $d=d'$. Then we merely have $(l,d)=(*,d)$ and $(l',d')=(*,d)$, so it is enough to note that the induced map on loop spaces based at $(*,d)$ is an equivalence by Proposition 1.7. Now let $d\neq d'$. To conclude we have to show $\mathcal{O}(k)$ is different from $\mathcal{O}(0)$ for $k\neq 0$. It is enough to show that for $k>0$ the bundle $\mathcal{O}(k)$ has at least two linear independent sections, since we know $\mathcal{O}(0)$ only has constant sections by Proposition 1.7. This follows from the fact that $\mathcal{O}(k)(x)$ is $\mathrm{Hom}_{R\text{-Mod}}(Rx^{\otimes k},R)$ and has all projections as sections. □

It is a curious remark that $K(R^\times,1)\to K(R^\times,1)$ is also equivalent to $K(R^\times,1)\times\mathrm{Hom}_{\mathrm{Group}}(R^\times,R^\times)=K(R^\times,1)\times\mathbb{Z}$.

Corollary 4.5 We have $\prod_{L:\mathbb{P}^1\to K(R^\times,1)}\prod_{x:R}L([1:x])=L([0:1])$.

Proof By the equivalence in Proposition 4.4, we have

$$\prod_{L:\mathbb{P}^1\to K(R^\times,1)}\ \prod_{x:\mathbb{P}^1}L(x)=l_0\otimes\mathcal{O}(d)(x)$$

for some (l_0,d) corresponding to L. $\mathcal{O}(d)([0:1])$ can be identified with R^1 and $\mathcal{O}(d)$ is trivial on R, so we have $L([1:x])=l_0=L([0:1])$ for all $x:R$. □

5 Line bundles on $\mathbb{P}^n$

We will prove $\mathrm{Pic}(\mathbb{P}^n)=\mathbb{Z}$ and a strengthening thereof in this section by mostly algebraic means. In Section 6 we will give a shorter geometric proof.

We can now reformulate Quillen's argument for Theorem 2' [Qui76] in our setting.

Proposition 5.1 For all $V:\mathbb{P}^n\to K(R^\times,1)$ we have $\prod_{s:R^n}V([1:s])=V([0:1:0:\dots:0])$.

Proof We define $L:R^{n-1}\to(\mathbb{P}^1\to K(R^\times,1))$ by $L\ t\ [x_0:x_1]=V([x_0:x_1:x_0t])$. Let $s=(s_1,\dots,s_n):R^n$. We apply Corollary 4.5 and we get

$$V([1:s])=L\ (s_2,\dots,s_n)\ [1:s_1]=L\ (s_2,\dots,s_n)\ [0:1]=V([0:1:0:\dots:0]).\qquad\square$$

Note that the use of Corollary 4.5 replaces the use of the "Quillen patching" [LQ15] introduced in [Qui76].

Let T be the ring of polynomials $u=\Sigma_p u(p)X^p$ with $X^p=X_0^{p_0}\dots X_n^{p_n}$ with $\Sigma p_i=0$. We write T_l for the subring of T which contains only monomials X^p with $p_i\geqslant 0$ if $i\neq l$ and T_{lm} the subring of T which contains only monomials X^p with $p_i\geqslant 0$ if $i\neq l$ and $i\neq m$.

Note that T_l is the polynomial ring $T_l=R[X_0/X_l,\dots,X_n/X_l]$.

A line bundle on $\mathbb{P}^n$ is given by compatible line bundles on each $\mathrm{Spec}(T_l)$.

By Proposition 5.1, a line bundle on $\mathbb{P}^n$ is trivial on each $\mathrm{Spec}(T_l)$. So it is determined by t_{ij} invertible in $T_i[X_i/X_j]=T_j[X_j/X_i]=T_{ij}$ such that $t_{ik}=t_{ij}t_{jk}$ and $t_{ii}=1$. Using Lemma 4.1 we can assume without loss of generality, that $t_{ij}=(X_i/X_j)^{N_{ij}}u_{ij}$, for some N_{ij} in $\mathbb{Z}$, where $u_{ij}(p)$ is invertible for $p=0$ and all other coefficients $u_{ij}(p)$ for $p\neq 0$ are nilpotent. By looking at the relation $t_{ik}=t_{ij}t_{jk}$ when we quotient by nilpotent elements, we see that $N_{ij}=N$ does not depend on i,j. The result $\mathrm{Pic}(\mathbb{P}^n)=\mathbb{Z}$ will then follow from the following result.

Proposition 5.2 There exists s_i invertible in T_i such that $u_{ij}=s_i/s_j$

Proof Each u_{ij} is such that $u_{ij}(p)$ unit for $p=0$ and all $u_{ij}(p)$ nilpotent for $p\neq 0$.

Like in the proof of Lemma 4.2, we can change u_{01} so that we have $u_{01}(p)=0$ if $p\neq 0$ and $p_0\geqslant 0$ or $p_1\geqslant 0$ by multiplying u_{01} by a unit in T_0 and a unit in T_1. Let us show for instance how to force $u_{01}(p)=0$ if $p\neq 0$ and $p_1\geqslant 0$ by multiplying u_{01} by a unit in T_0. Let M be the ideal generated by $u_{01}(p)$ for $p\neq 0$, which is a nilpotent ideal. If we multiply u_{01} by $u_{01}(0)-\Sigma_{p_1\geqslant 0}u_{01}(p)$ we change u_{01} to

u'_{01} where all $u'_{01}(p)$, for $p_1 \geqslant 0$ and $p \neq 0$, are in M^2. We iterate this process and since M is nilpotent, we force $u_{01}(p) = 0$ or $p \neq 0$ and $p_1 \geqslant 0$.

We can thus assume that $u_{01}(p) = 0$ if $p \neq 0$ and $p_0 \geqslant 0$ or $p_1 \geqslant 0$.

We claim then that, in this case, u_{01} has to be a unit. For this we show that $u_{01}(p) = 0$ if $p_l > 0$ for each $l \neq 0, 1$. This is obtained by looking at the relation $u_{01} = u_{0l}u_{l1}$. Let L be the ideal generated by coefficients $u_{0l}(p)$ and $u_{1l}(p)$ with $p_l > 0$ and I the ideal generated by all nilpotent coefficients of u_{0l} and u_{l1}. Thanks to the form of u_{01} we must have $L \subseteq LI$ and so $L = 0$ by Nakayama. Indeed we have

$$u_{01}(p) = u_{0l}(p)u_{l1}(0) + u_{0l}(0)u_{l1}(p) + \Sigma_{q+r=p,q\neq 0,r\neq 0}u_{0l}(q)u_{l1}(r)$$

and we use this to show that $u_{0l}(p)$ is in LI. Since $p_l > 0$, we have $u_{0l}(p) = 0$ if $p_0 \geqslant 0$, hence we can assume $p_0 < 0$. We also have $u_{0l}(p)$ if $p_1 < 0$ and we can assume $p_1 \geqslant 0$. This implies $u_{l1}(p) = 0$ (since $p_0 < 0$) and $u_{01}(p) = 0$ (since $p_0 < 0$ and $0 \leqslant p_1$). We get thus

$$u_{0l}(p)u_{l1}(0) = -\Sigma_{q+r=p,q\neq 0,r\neq 0}u_{0l}(q)u_{l1}(r)$$

and each member in the sum $u_{0l}(q)u_{l1}(r)$ is in IL since $q_l + r_l = p_l > 0$ and hence $q_l > 0$ or $r_l > 0$.

We thus deduce $L = 0$ by Nakayama. We get, for $p_l > 0$

$$u_{01}(p) = u_{0l}(p)u_{l1}(0) + u_{0l}(0)u_{l1}(p)$$

and if $p_0 < 0$ and $p_1 < 0$ we have $u_{0l}(p) = u_{l1}(p) = 0$.

This implies that all coefficients $u_{01}(p)$ such that $p_l > 0$ are 0.

Since this holds for each $l > 1$ we have that u_{01} is a unit in R.

W.l.o.g. we can assume $u_{01} = 1$. We then have $u_{0l} = u_{1l}$ in $T_{0l} \cap T_{1l} = T_l$ and we take $s_l = u_{0l} = u_{1l}$.□

Corollary 5.3 $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$.

We can then strengthen this result, with the same reasoning as in Proposition 4.4.

Theorem 5.4
The map $K(R^\times, 1) \times \mathbb{Z} \to (\mathbb{P}^n \to K(R^\times, 1))$ which associates to l_0, d the map $x \mapsto l_0 \otimes \mathcal{O}(d)(x)$ is an equivalence.

We deduce from this a characterisation of the maps $\mathbb{P}^n \to \mathbb{P}^m$.

Corollary 5.5 A map $\mathbb{P}^n \to \mathbb{P}^m$ is given by $m+1$ homogeneous polynomials $p = (p_0, \ldots, p_m)$ on R^{n+1} of the same degree d such that $x \neq 0$ implies $p(x) \neq 0$.

Proof Write $T_n(l)$ for $l^{n+1} \setminus \{0\}$. We have $\mathbb{P}^n = \Sigma_{l:K(R^\times,1)}T_n(l)$ and so

$$\mathbb{P}^n \to \mathbb{P}^m = \sum_{s:\mathbb{P}^n \to K(R^\times,1)} \prod_{x:\mathbb{P}^n} T_m(s\ x)$$

Using Theorem 5.4, this is equal to

$$\sum_{l_0:K(R^\times,1)} \sum_{d:\mathbb{Z}} \prod_{l:K(R^\times,1)} T_n(l) \to T_m(l_0 \otimes l^{\otimes d})$$

and, as for Lemma 1.9, this is the set of tuples of $m+1$ polynomials in $R[X_0, \ldots, X_n]$ homogenenous of degree d, sending $x \neq 0$ to $p(x) \neq 0$, and quotiented by proportionality. □

We deduce the characterisation of $\mathrm{Aut}(\mathbb{P}^n)$. This is a remarkable result, since the automorphisms are in this framework only bijections of sets.

Corollary 5.6 $\mathrm{Aut}(\mathbb{P}^n)$ is PGL_{n+1}.

We also have the following application of computation of cohomology groups [CCH23].

Corollary 5.7 A function $\mathbb{P}^n \to \mathbb{P}^m$ is constant if $n > m$.

Proof We proved in [BCW23] that cohomology groups can be computed as Cech cohomology for any finite open acyclic covering and used this to prove $H^n(\mathbb{P}^n, \mathcal{O}(-n-1)) = R$. By Corollary 5.5, a map $\mathbb{P}^n \to \mathbb{P}^m$ is given by $m+1$ non zero polynomials $p(x) = (p_0(x), \ldots, p_m(x))$ homogeneous of the same degree $d \geqslant 0$ and such that $x \neq 0$ implies $p(x) \neq 0$. This means that $\mathbb{P}^n$ is covered by $m+1$ open subsets $U_i(x)$ defined by $p_i(x) \neq 0$. I claim that we should have $d = 0$.

If $q(x)$ is a non zero homogeneous polynomial of degree $d > 0$, the open $q(x) \neq 0$ defines an *affine* and hence acyclic [BCW23], open subset of $\mathbb{P}^n$ (see e.g. Exercise 3.5 in [Har77]). It follows that the covering $U_0, \ldots, U_m$ is acyclic if $d > 0$. But this contradicts $H^n(\mathbb{P}^n, \mathcal{O}(-n-1)) = R$.

Hence $d = 0$ and the map is constant. □

6 A geometric proof of $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$

A geometric property of $\mathbb{P}^n$:

Lemma 6.1 Let $n > 1$ and $p \neq q$ be points of $\mathbb{P}^n$, then all functions

(i) $\mathbb{P}^n \setminus \{p\} \to \mathbb{Z}$

(ii) $\mathbb{P}^n \setminus \{p, q\} \to \mathbb{Z}$

(iii) $\mathbb{P}^n \setminus \{p\} \to R$

(iv) $\mathbb{P}^n \setminus \{p, q\} \to R$

are constant.

Proof We start with (iv). Let $f : \mathbb{P}^n \setminus \{p, q\} \to R$. For the charts $U_0 = \{[x_0 : \cdots : x_n] \mid x_0 \neq 0\}$ and $U_1 = \{[x_0 : \cdots : x_n] \mid x_0 \neq 0\}$, we can assume $p \in U_0, p \notin U_1$ and $q \in U_1, q \notin U_0$. Then $f_{|U_0 \setminus \{p\}}$ can be extended to U_0 by Lemma 1.6 and an analogous extension exisits on U_1. These extensions glue with f to a function $\widetilde{f} : \mathbb{P}^n \to R$ which agrees with f on $\mathbb{P}^n \setminus \{p, q\}$. By Proposition 1.7, $\widetilde{f}$ is constant and therefore f is constant. This carries over to functions $\mathbb{P}^n \setminus \{p, q\} \to \mathrm{Bool}$ since $\mathrm{Bool} \subseteq R$ and thus also to any $\mathbb{P}^n \setminus \{p, q\} \to \mathbb{Z}$, which shows (ii). (i) and (iii) follow from (ii) and (iv). $\square$

We proceed by extendending Proposition 4.4 to subspaces of $\mathbb{P}^n$ which can be constructed like $\mathbb{P}^1$:

Lemma 6.2 Let $M \subseteq R^{n+1}$ be a submodule with $\|M = R^2\|$. Then $\mathrm{Gr}(1, M) \subseteq \mathbb{P}^n$ and the map

$$\begin{aligned} \mathbb{Z} \times K(R^\times, 1) &\to (\mathrm{Gr}(1, M) \to K(R^\times, 1)) \\ (d, l_0) &\mapsto (L \mapsto L^{\otimes d}) && \text{for } d \geq 0 \\ (d, l_0) &\mapsto (L \mapsto \mathrm{Hom}_{R\text{-Mod}}(L^{\otimes d}, R)) && \text{for } d < 0 \end{aligned}$$

is an equivalence.

Proof We prove a proposition, so we have an R-linear isomorphism $\phi : R^2 \to M$ and for each $d : \mathbb{Z}$, we get a commutative triangle:

$$\begin{tikzcd} \mathrm{Gr}(1, M) \arrow[rr, "\mathcal{O}(d)"] & & K(R^\times, 1) \\ & \mathrm{Gr}(1, R^2) \arrow[ul, "\phi"] \arrow[ur, "\mathcal{O}(d)"'] & \end{tikzcd}$$

by restricting ϕ to each line in $\mathrm{Gr}(1, R^2)$. This shows that the map from Proposition 4.4 and from the statement are equal as maps to $(V : R\text{-Mod}) \times \|V = R^2\| \times (V \to K(R^\times, 1)))$, which proves the claim. $\square$

Theorem 6.3
The map

$$\begin{aligned} \mathbb{Z} \times K(R^\times, 1) &\to (\mathbb{P}^n \to K(R^\times, 1)) \\ (d, l_0) &\mapsto (x \mapsto l_0 \otimes \mathcal{O}(d)(x)) \end{aligned}$$

is an equivalence.

Proof It is enough to show that the map is surjective, by the same reasoning as in the proof of Proposition 4.4. Let $L : \mathbb{P}^n \to K(R^\times, 1)$. First we determine the degree of L. Let $p \neq q$ be points in $\mathbb{P}^n$ and $M \subseteq R^{n+1}$ be the span of p and q as submodules of R^{n+1}. Then $\|M = R^2\|$ and we can use the inverse i of the map in Lemma 6.2 to define $d :\equiv \pi_1(i(L_{|\mathrm{Gr}(1,M)}))$. The integer d is independent of the choice of p and q: If we let p vary, we get a function of type $\mathbb{P}^n \setminus \{q\} \to \mathbb{Z}$ which is constant by Lemma 6.1. The same applies for q and the two subsets $\mathbb{P}^n \setminus \{p\}$ and $\mathbb{P}^n \setminus \{q\}$ cover $\mathbb{P}^n$.

In the following we consider only L such that d as constructed above is 0. This means that on each line $\mathrm{Gr}(1, M)$, L will be constant. So for $p, x : \mathbb{P}^n$, and $x \neq p$ we can construct an equality $P_x : L(x) = L(p)$ by restricting L to $\mathrm{Gr}(1, \langle x, p \rangle)$ and applying Lemma 6.2. So we have $P : (x : \mathbb{P}^n \setminus \{p\}) \to L(x) = L(p)$ and for $q \neq p$ we can construct $Q : (y : \mathbb{P}^n \setminus \{q\}) \to L(y) = L(q)$ analogously.

The claim follows if we show that L is constant on all of $\mathbb{P}^n$. Since, overall, we show the proposition that the map from the statement merely has a preimage, we can assume $a : L(p) = R^1$ and $b : L(q) = R^1$ and get:

$$\big((x : \mathbb{P}^n \setminus \{p, q\}) \mapsto a^{-1} P_x^{-1} Q_x b\big) : \mathbb{P}^n \setminus \{p, q\} \to R^\times$$

which is constantly λ by Lemma 6.1. So P and Q can be corrected using λ, a and b to yield a global proof of constancy of L. □

Appendix 1: Quillen Patching

We reproduce the argument in Quillen's paper [Qui76], as simplified in [LQ15]. This technique of Quillen Patching has been replaced by the equivalence in Proposition 4.4.

If P and Q are two idempotent matrix of the same size, let us write $P \simeq Q$ for expressing that P and Q presents the same projective module (which means that there are similar, which is in this case is the same as being equivalent).

If we have a projective module on $A[X]$, presented by a matrix $P(X)$, this module is extended precisely when we have $P(X) \simeq P(0)$.

Lemma 6.4 If S is a multiplicative monoid of A and $P(X) \simeq P(0)$ on $A_S[X]$ then there exists s in S such that $P(X + sY) \simeq P(X)$ in $A[X]$.

Lemma 6.5 The set of s in A such that $P(X + sY) \simeq P(X)$ is an ideal of A.

Corollary 6.6 If we have M projective module of $A[X]$ and $S_1, \ldots, S_n$ comaximal multiplicative monoids of A such that each $M \otimes_{A[X]} A_{S_i}[X]$ is extended from A_{S_i} then M is extended from A.

Let us reformulate in synthetic term this result. Let A be a f.p. R-algebra and $L : \mathrm{Spec}(A) \to B\mathbb{G}_m^{\mathbb{A}^1}$. Then L corresponds to a projective module of rank 1 on $A[X]$. We can form

$$T(x) = \prod_{r:R} L\ x\ r = L\ x\ 0$$

and $\|T(x)\|$ expresses that $L\ x$ defines a trivial line bundle on $\mathbb{A}^1 = \mathrm{Spec}(R[X])$. It is extended exactly when we have $\|\prod_{x:\mathrm{Spec}(A)} T(x)\|$. We can then use Zariski local choice to state.

Proposition 6.7 We have the implication $(\prod_{x:\mathrm{Spec}(A)} \|T(x)\|) \to \| \prod_{x:\mathrm{Spec}(A)} T(x)\|$.

Appendix 2: Classical argument

We reproduce a message of Brian Conrad in MathOverflow [Con].

"We know that the Picard group of projective $(n-1)$-space over a field k is $\mathbb{Z}$ generated by $\mathcal{O}(1)$. This underlies the proof that the automorphism group of such a projective space is $\mathrm{PGL}_n(k)$. But what is the automorphism group of $\mathbb{P}^{n-1}(A)$ for a general ring A? Is it $\mathrm{PGL}_n(A)$? It's a really important fact that the answer is yes. But how to prove it? It's a shame that this isn't done in Hartshorne.

By an elementary localization, we may assume A is local. In this case we claim that $\mathrm{Pic}(\mathbb{P}^{n-1}(A))$ is infinite cyclic generated by $\mathcal{O}(1)$. Since this line bundle has the known A-module of global sections, it would give the desired result if true by the same argument as in the field case. And since we know the Picard group over the residue field, we can twist to get to the case when the line bundle is trivial on the special fiber. How to do it?

Step 0: The case when A is a field. Done.

Step 1: The case when A is Artin local. This goes via induction on the length, the case of length 0 being Step 0 and the induction resting on cohomological results for projective space over the residue field.

Step 2: The case when A is complete local noetherian ring. This goes using Step 1 and the theorem on formal functions (formal schemes in disguise).

Step 3: The case when A is local noetherian. This is faithfully flat descent from Step 2 applied over $A^\frown$

Step 4: The case when A is local: descent from the noetherian local case in Step 3 via direct limit arguments.

QED"

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