Logical Topology and Axiomatic Cohesion

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Axiomatic Cohesion – A Refresher

- Lawvere proposes to continue the following dialogue:

  "What is a space?"
  "It is an object of a category of spaces."
  "Then what is a category of spaces?"

- Lawvere's *wu wei* axiomatization of "space": modalities that remove all "spatial cohesion" in three different ways.
  - $\sharp$: whose modal types are the codiscrete spaces.
  - $\flat$: whose modal types are the discrete spaces.
  - $\int$: whose modal types are the discrete spaces (but whose action is different).

Models of Cohesion

Some *gros topoï* of interest are cohesive toposes:

- **Continuous Sets** as in Shulman's *Real Cohesion*.
- **Dubuc's Topos** and **Formal Smooth Sets** as in Synthetic Differential Geometry and Schreiber's *Differential Cohesion*.
- **Menni's Topos** (similar to the big **Zariski Topos**) as in algebraic geometry.*

In all of these models, there are suitably nice spaces

- continous manifolds,
- smooth manifolds,
- (suitable) schemes,

which have topologies (via open sets) on their underlying sets.

Penon's Logical Topology

In his thesis, Penon defined a **Logical Topology** held by any type.

Definition (Penon)

A subtype $U : A \to$ **Prop** is **logically open** if

- For all $x, y : A$ with x in U, either $x \neq y$ or y is in U.

Penon and Dubuc proved that in the three examples

- **Continuous Sets**: Logical opens on continous manifolds are ϵ-ball opens.
- **Dubuc's Topos**: Logical opens on smooth manifolds are ϵ-ball opens.
- **Zariski Topos**: Logical opens on (suitable) separable schemes are Zariski opens.

Motivating Question:

How does the logical topology on a type compare with its cohesion?

We will see two glimpses today:

- The path connected components $\int_0 A$ (defined through cohesion) are the same as the logically connected components of A.
- A set is **Leibnizian** (defined through cohesion) if and only if it is de Morgan (a logical notion).

Cohesive Type Theory Refresher

In his *Real Cohesion*, Shulman gave a type theory for axiomatic cohesion. Cohesive type theory uses two kinds of variables:

- Cohesive variables, which vary "continuously".
- Crisp variables, which vary "discontinuously".

Following Shulman, we assume the following:

Axiom (LEM)

If $P :: \mathbf{Prop}$ is a crisp proposition, then either P or $\neg P$ holds.

Every discontinuous proposition is either true or false.

Cohesive Type Theory Refresher

We will also assume that $\int$ is given by nullifying some "basic contractible space(s)".

Axiom (Punctual Local Contractibility)

There is a type $\mathbb{A}$ :: **Type** such that:

- A crisp type X is discrete if and only if it is homotopical – the inclusion of constants $X \to (\mathbb{A} \to X)$ is an equivalence, and
- There is a point $0 :: \mathbb{A}$ in each of these types.

We can consider a map $\gamma : \mathbb{A} \to X$ to be a *path* in X.

- This means that $\int A$ is the **homotopy type** (or **fundamental** ∞-**groupoid**) of A, considered as a discrete type.
- And, therefore,

$$\int_0 A :\equiv \|\int A\|_0$$

is the set of path connected components of A.

Path components = Connected components?

- So,

$$\int_0 A :\equiv \|\int A\|_0$$

  is the set of path connected components of A.
- Is it also the set of *logical* connected components of A?

The Powerset of a Type

Definition

Given a type A, its *powerset* $\mathcal{P}\, A :\equiv A \to \mathbf{Prop}$ is the set of propositions depending on an $a : A$. The order on subtypes is given by:

$$P \subseteq Q :\equiv \forall a.\, Pa \Rightarrow Qa$$

We define the usual operations on subtypes point-wise:

$$P \cap Q :\equiv \lambda a.\, Pa \wedge Qa$$
$$P \cup Q :\equiv \lambda a.\, Pa \vee Qa$$
$$\neg P :\equiv \lambda a.\, \neg Pa$$

Logical Connected Components

Definition

1. A subtype $U : \mathcal{P}\, A$ is *merely inhabited* if there is merely an $a : A$ such that Ua.
2. A subtype $U : \mathcal{P}\, A$ is *detachable* if for all $a : A$, Ua or $\neg Ua$.
3. A subtype $U : \mathcal{P}\, A$ is *logically connected* if for all $P : \mathcal{P}\, A$, if $U \subseteq P \cup \neg P$, then $U \subseteq P$ or $U \subseteq \neg P$.

Definition

A subtype $U : \mathcal{P}\, A$ is a *logical connected component* if it is merely inhabited, detachable, and logically connected.

Lemma

If U and V are logical connected components of A, and $U \cap V$ is non-empty, then $U = V$.

$\int_0$ gives the Logical Connected Components

We let $\int_0 A :\equiv \|\int A\|_0$, and $\sigma_0 : A \to \int_0 A$ be its unit.

Lemma

For any type A and any $u : \int_0 A$, the proposition $\sigma_0^* u :\equiv \lambda a.\, \sigma_0 a = u$ is a logical connected component of A.

Proof.

- $\sigma_0^* u$ is merely inhabited because σ_0 is merely surjective (PLC).
- Since $\int_0 A$ is a discrete set, it has decideable equality (LEM). Therefore, $\sigma_0^* u$ is detachable.
- If $\sigma_0^* u \subseteq P \cup \neg P$, then we can define $\bar{P} : (a : A) \times \sigma_0^* u(a) \to \{0, 1\}$ by cases. But $(a : A) \times \sigma_0^* u(a) \equiv \mathrm{fib}_{\sigma_0}(u)$ and so is $\int_0$-connected; therefore, $\bar{P}$ is constant, and $\sigma_0^* u \subseteq P$ or $\sigma_0^* u \subseteq \neg P$.

□

$\int_0$ gives the Logical Connected Components

Theorem

For a type A, the map σ_0^* gives an equivalence between $\int_0 A$ and the set of logical connected components of A.

Infinitesimals and Double Negation

In his paper *Infinitesimaux et Intuitionisme*, Penon makes the following claims:

Proposition (Kock)

In the big Zariski or étale topos, with $\mathbb{A}$ the affine line,

$$\neg\neg\{0\} = \mathrm{Spec}(\mathbb{Z}[[t]]) = \{a : \mathbb{A} \mid \exists n.\, a^n = 0\}$$

is the set of nilpotent infinitesimals.

Proposition (Penon)

In Dubuc's topos, with $\mathbb{A}$ the sheaf co-represented by $\mathcal{C}^\infty(\mathbb{R})$,

$$\neg\neg\{0\} = よ(\mathcal{C}^\infty_0(\mathbb{R}))$$

is co-represented by the germs of smooth functions at 0.

Ainsi donc l'écriture

$$\neg\neg\{0\} = \{\text{ Infinitésimaux }\}$$

est justifiée.

Neighbors and Germs

Definition

Let A : **Type**, and let $a, b : A$. We say a and b are **neighbors** if they are not distinct:

$$a \approx b :\equiv \neg\neg(a = b).$$

Proposition

The neighboring relation is reflexive, symmetric, and transitive, and is preserved by any function $f : A \to B$.

- For $a : A$, $a \approx a$,
- For $a, b : A$, $a \approx b$ implies $b \approx a$,
- For $a, b, c : A$, $a \approx b$ and $b \approx c$ imply $a \approx c$,
- For $a, b : A$ and $f : A \to B$, if $a \approx b$, then $f(a) \approx f(b)$.

Neighbors and Germs

Definition

The **neighborhood** $\mathbb{D}_a$ of $a : A$ is the type of all its neighbors:

$$\mathbb{D}_a :\equiv (b : A) \times a \approx b.$$

The **germ** of $f : A \to B$ at $a : A$ is

$$\begin{aligned} df_a &: \mathbb{D}_a \to \mathbb{D}_{f(a)} \\ (d, \_) &\mapsto (f(d), \_) \end{aligned}$$

Proposition

(Chain rule) For $f : A \to B$, $g : B \to A$, and $a : A$,

$$d(g \circ f)_a = dg_{f(a)} \circ df_a.$$

Cohesion Refresher

Theorem (Shulman)

$\sharp$ is lex: for any $x, y : A$, there is an equivalence $(x^\sharp = y^\sharp) \simeq \sharp(x = y)$ such that the following diagram commutes.

$$\begin{array}{ccc} & & x^\sharp = y^\sharp \\ & \overset{\mathsf{ap}_{(-)^\sharp}}{\nearrow} & \\ x = y & & \big\downarrow \simeq \\ & \underset{(-)^\sharp}{\searrow} & \\ & & \sharp(x = y) \end{array}$$

Lemma (Shulman)

For any P : **Prop**, $\sharp P = \neg\neg P$, and a proposition is codiscrete if and only if it is not-not stable.

Codiscretes and Infinitesimals

Putting these facts together, we get:

Proposition

For a set A and points $a, b : A$,

$$a \approx b \quad \equiv \quad \neg\neg(a = b) \iff \sharp(a = b) \iff a^\sharp = b^\sharp$$

Corollary

0 is the only crisp infinitesimal.

In fact, since

$$\begin{aligned}\mathrm{fib}_{(-)^\sharp}(x^\sharp) &:\equiv (y : A) \times x^\sharp = y^\sharp \\ &\simeq (y : A) \times x \approx y \equiv: \mathbb{D}_x\end{aligned}$$

we have that all formal discs $\mathbb{D}_x$ are $\sharp$-connected.

Leibnizian Sets and the Leibniz Core

Definition (Lawvere)

A set A is *Leibnizian* if $\sharp\sigma : \sharp A \to \sharp \int A$ is an equivalence, where $\sigma : A \to \int A$ is the unit.
For crisp sets, this is equivalent to the *points-to-pieces* transform $\sigma \circ (-)_\flat : \flat A \to \int A$ being an equivalence.

Every piece contains exactly one crisp point.

Definition

The *Leibniz core* $\mathcal{L}\, A$ of a crisp set A is the pullback

$$\begin{aligned}\mathcal{L}\, A &:\equiv (a : \flat A) \times (b : A) \times {a_\flat}^\sharp = b^\sharp \\ &\simeq (a : \flat A) \times \mathbb{D}_{a_\flat}\end{aligned}$$

A Set is Leibnizian if and only if it is de Morgan

Definition

A type A is *de Morgan* if for all $a, b : A$,

$$a \approx b \quad \text{or} \quad a \not\approx b.$$

Theorem

A set A is Leibnizian if and only if it is de Morgan

Compare with:

Theorem (Shulman)

A set A is discrete if and only if it is decidable – that is,

$$\text{for } a, b : A, \ a = b \text{ or } a \neq b.$$

Sketching a Proof

Theorem

A set A is Leibnizian if and only if it is de Morgan

If A is Leibnizian, then $\sharp\sigma_0$ is an equivalence as well. For a, $b : A$, either $\sigma_0 a = \sigma_0 b$ or not; therefor, $({}_0 a)^\sharp = (\sigma_0 b)^\sharp$ or not. Naturality then gives us that $\sharp\sigma_0(a^\sharp) = \sharp\sigma_0(b^\sharp)$ or not. But $\sharp\sigma_0$ is an equivalence, so $a^\sharp = b^\sharp$ or not.

On the other hand, if A is de Morgan we can give an inverse to $\sharp$ by sending $u : \sharp \int A$ to $x^\sharp$ where $\sigma x = u_\sharp$. This is well defined since we can map $y : \mathrm{fib}_\sigma(\sigma x)$ to $\{0, 1\}$ according to whether or not $y \approx x$; this shows that every y in the fiber of σx is its neighbor, and therefore that $y^\sharp = x^\sharp$.

References

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