Degrees, Dimensions, and Crispness

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Outline

- The upper naturals.
- The algebra of polynomials, three ways.
- Crisp things have natural number degree / dimension.
The Logic of Space

Space-y-ness of your domains of discourse

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Constructiveness of the (native) logic about things in those domains
Logical Connectivity

Definition

A proposition $U : A \rightarrow \text{Prop}$ is **logically connected** if for all $P : A \rightarrow \text{Prop}$, if $\forall a. Ua \rightarrow Pa \lor \neg Pa$, then either $\forall a. Ua \rightarrow Pa$ or $\forall a. Ua \rightarrow \neg Pa$.

Lemma

If $U : A \rightarrow \text{Prop}$ is logically connected and $f : A \rightarrow B$, then its image $\text{im}(U) \equiv \lambda b. \exists a. f(a) = b \land Ua : B \rightarrow \text{Prop}$ is logically connected.

Lemma

If $A$ has decidable equality (either $a = b$ or $a \neq b$), then a logically connected $U : A \rightarrow \text{Prop}$ has at most one element.
Degree of a Polynomial

- Suppose $R$ is a ring. Naively, taking the degree of a polynomial should give a map
  $\deg : R[x] \to \mathbb{N}$

- But suppose that $R$ is logically connected and for $r : R$ consider the polynomial $rx$.

- Then $\deg(rx) : \mathbb{N}$, so that
  $\lambda r. \deg(rx) : R \to \mathbb{N}$.

  But $R$ is connected and $\mathbb{N}$ has decidable equality, so this map must be constant (by the lemma).

- Of course, $\deg(x) = 1$ and $\deg(0) = 0$, so this proves $1 = 0$, which is an issue.
Problems with the Naturals

So there’s a problem with the naturals – they are too _discrete_. How do we fix this?

To solve this, we need to find another problem with the natural numbers: one from _logic_.

**Proposition**

The law of excluded middle (LEM) is equivalent to the well-ordering principle (WOP) for $\mathbb{N}$.

**Proof.**

That the classical naturals satisfy WOP is routine. Let’s show that the well-ordering of $\mathbb{N}$ implies LEM.

Given a proposition $P : \text{Prop}$, define $\bar{P} : \mathbb{N} \rightarrow \text{Prop}$ by $\bar{P}(n) :\equiv P \lor 1 \leq n$ and note that $\bar{P}(0) = P$. The least number satisfying $\bar{P}$ is 0 or not depending on whether $P$ or $\neg P$; since equality of naturals is decidable, either $P$ or $\neg P$. 
The Upper Naturals

In other words,

*The naturals are not complete as a Prop-category.*

So, let’s freely complete them! We will replace a natural number $n : \mathbb{N}$ by its upper bounds $\lambda m. n \leq m : \mathbb{N} \to \text{Prop}$.

**Definition**

The **upper naturals** $\mathbb{N}^\uparrow$ are the type of upward closed propositions on the naturals. (As a Prop-category, this is $(\text{Prop}^\mathbb{N})^{\text{op}}$)

- We think of an upper natural $N : \mathbb{N}^\uparrow$ as a natural “defined by its upper bounds”:
  
  $$Nn \text{ holds if } n \text{ is an upper bound of } N.$$  

- For $N, M : \mathbb{N}^\uparrow$, say $N \leq M$ when every upper bound of $M$ is an upper bound of $N$. 
Naturals and Upper Naturals

Definition

The **upper naturals** $\mathbb{N}^{\uparrow}$ are the type of upward closed propositions on the naturals.

Every natural $n : \mathbb{N}$ gives an upper natural $n^{\uparrow} : \mathbb{N}^{\uparrow}$ by the Yoneda embedding:

$$n^{\uparrow}(m) :\equiv n \leq m.$$ and we define $\infty^{\uparrow} :\equiv \lambda_\perp. \text{False}.$

An upper natural $N : \mathbb{N}^{\uparrow}$ is **bounded** if there exists an upper bound $n : \mathbb{N}$ of $N$ (that is, if $\exists n. Nn$).

We can take the minimum upper natural satisfying a proposition:

$$\text{min} : (\mathbb{N} \to \text{Prop}) \to \mathbb{N}^{\uparrow}$$

by

$$(\text{min } P)n :\equiv \exists m \leq n. Pm$$
Upper Arithmetic

Definition

\[ \text{min} : (\mathbb{N} \rightarrow \text{Prop}) \rightarrow \mathbb{N}^\uparrow \]

\[ P \mapsto \lambda n. \exists m \leq n. Pm \]

Lemma

For \( P : \mathbb{N} \rightarrow \text{Prop} \), \( \text{min } P = n^\uparrow \) if and only if \( n \) is the least number satisfying \( P \).

We can define the arithmetic operations for upper naturals by Day convolution: (with \( N, M : \mathbb{N}^\uparrow \))

- \( (N + M)n \equiv \exists a, b : \mathbb{N} . Na \land Mb \land (a + b \leq n) \).
- \( (N \cdot M)n \equiv \exists a, b : \mathbb{N} . Na \land Mb \land (ab \leq n) \).
- And one can prove the expected identities by the usual Day convolution arguments.
Upper Naturals in Models

- In localic models, $\mathbb{N}^\uparrow$ is the sheaf of upper semi-continuous functions valued in $\mathbb{N}$.
- (Hartshorne (1977) Example III.12.7.2) If $Y$ is a Noetherian scheme and $\mathcal{F}$ a coherent sheaf of modules on $Y$, then
  
  $$y \mapsto \dim_{k(y)}(\mathcal{F}_y \otimes k(y))$$

  is an upper-semicontinuous function $Y \to \mathbb{N}$, and therefore a global section of $\mathbb{N}^\uparrow \in \mathbf{Sh}(Y)$.
- For more on the upper naturals in a localic setting, see Section II.5 of Blechschmidt (2017). (There they are called generalized naturals)
Cardinality

As an example of what we can define with upper naturals that we couldn’t with naturals, consider:

Definition

Define the (finite) **cardinality** of a type as

\[
\text{Card} : \text{Type} \rightarrow \mathbb{N}^\uparrow
\]

\[
X \mapsto \min (\lambda n. \| [n] \simeq X \| )
\]

(or, the Kuratowski cardinality by \( X \mapsto \min (\lambda n. \exists f : [n] \rightarrow X) \))

Proposition

We have the expected equations:

- \( \text{Card}(X + Y) = \text{Card}(X) + \text{Card}(Y). \)
- \( \text{Card}(X \times Y) = \text{Card}(X) \cdot \text{Card}(Y). \)
- \( \text{Card}(X +_{U} Y) = \text{Card}(X) + \text{Card}(Y) - \text{Card}(U). \)*
Polynomials, Three Ways

To define the degree of a polynomial, we need to define the algebra of polynomials. In the following, let $R$ be a ring.

**Definition**

For a type $I$, the free $R$-algebra on $I$, $R[x_i \mid i : I]$ is the higher inductive type generated by

- $x : I \to R[x_i \mid i : I]$
- struct : $R$-algebra structure on $R[x_i \mid i : I]$

**Proposition**

Let $A$ be an $R$-algebra and $I$ a type. Then evaluating at $x : I \to R[x_i \mid i : I]$ gives an equivalence

\[(I \to A) \simeq \text{Alg}_R(R[x_i \mid i : I], A).\]
Polynomials, Three Ways

This gives a straightforward definition of $R[x]$ as $R[x_i \mid i : \ast]$.

But it’s not immediately clear how to define the degree of a polynomial using this definition. Let’s give another:

**Definition**

Define $R[x]^s$ to be the type of eventually vanishing sequences in $R$. That is

$$R[x]^s \equiv (f : \mathbb{N} \to R) \times \exists n. \forall m > n. f_m = 0.$$ 

**Proposition**

Let $A$ be an $R$-algebra. Then, evaluation at $x : R[x]^s$ gives an equivalence

$$A \simeq \text{Alg}_R(R[x]^s, A).$$
The Degree of a Polynomial

Now we can define

$$\deg : R[x]^s \rightarrow \mathbb{N}^\uparrow$$

$$\deg(f)n \equiv: \forall m > n. f_m = 0$$

We can prove some basic facts about the degree:

- If $$\deg(f) = n^\uparrow$$, then $$f = \sum_{i=0}^n f_i x^i$$.
- $$\deg(f + g) \leq \max\{\deg(f), \deg(g)\}.$$  
- $$\deg(fg) \leq \deg(f) + \deg(g).$$
- What about $$\deg(f \circ g) \leq \deg(f) \cdot \deg(g)$$?
Horner Normal Form

We note that any polynomial $f$ can be written as

$$f(x) = g(x) \cdot x + f(0)$$

Definition

Let $R[x]^h$ be the higher inductive type given by

- $\text{const} : R \to R[x]^h$,
- $(-) \cdot x + (-) : R[x]^h \times R \to R[x]^h$,
- $\text{eq} : (r : R) \to \text{const}(0) \cdot x + \text{const}(r) = \text{const}(r)$,
- $\text{is-set} : R[x]^h$ is a set.

Proposition

For any $R$-algebra $A$, evaluation at $\text{const}(1) \cdot x + \text{const}(0)$ gives an equivalence

$$A \simeq \text{Alg}_R(R[x]^h, A).$$
Definition

Define the composite \( f \circ g \) of two polynomials \( f, g : R[x]^h \) by induction on \( f \):

- If \( f \equiv \text{const}(r) \), then \( f \circ g \equiv \text{const}(r) \).
- If \( f \equiv h \cdot x + \text{const}(r) \), then \( f \circ g \equiv (h \circ g) \cdot g + \text{const}(r) \).
- We check that \( (0 \cdot x + r) \circ g = r \), and
- We note we are mapping into a set.
Induction on Degree Horner Normal Form

**Proposition**

For any polynomials $f, g : R[x]^h$, $\deg(f \circ g) \leq \deg(f) \cdot \deg(g)$.

**Proof.**

By induction on horner normal form:

$$\deg((f(x)x + r) \circ g) = \deg((f \circ g)(x) \cdot g(x) + r)$$

$$= \deg((f \circ g)(x) \cdot g(x))$$

$$\leq \deg((f \circ g)) + \deg(g)$$

$$\leq \deg(f) \cdot \deg(g) + \deg(g) \quad \text{by hypothesis}$$

$$= (\deg(f) + 1^\uparrow) \cdot \deg(g)$$

$$= \deg(f(x) \cdot x + r) \cdot \deg(g)$$
Slogan: Instead of inducting on degree, induct on the polynomial!
Definition

We define the dimension of a vector space $V$ over a field $k$ by

$$(\dim V)n \equiv \min(\lambda n. \|k^n \cong V\|)$$

It is the minimum $n$ such that $V$ has an $n$-element basis

Proposition

Let $f : k[x]$. Then $\deg(f) = \dim(k[x]/(f))$. 
Catching up on Crispness

- Recall that Shulman’s cohesive homotopy type theory uses crisp variables to keep track of discontinuous dependency. A term is crisp if all the free variables in it are crisp.
- Crisp variables must have crisp type, and only crisp terms can be substituted for crisp variables.
- So, $x :: X$ – a crisp point of $X$ – is a general discontinuous element of $X$.

**Axiom (LEM)**

If $P :: \text{Prop}$ is a crisp proposition, then either $P$ or $\neg P$ holds.

*Discontinuously, every proposition is either true or false.*
Crisp upper naturals are extended naturals

- If \( X \) is a crisp type, then \( bX \) can be thought of as the type of crisp points of \( X \).

**Definition**

The **Extended Naturals** \( \mathbb{N}^\infty \) is the type of monotone functions \( \mathbb{N} \rightarrow \text{Bool} \). Equivalently, it is the type of upwards-closed *decidable* propositions on the naturals.

**Proposition**

- The extended naturals embed into the upper naturals, preserving the naturals.
- The bounded extended naturals are equivalent to the naturals. Every decidable, inhabited subset of \( \mathbb{N} \) has a least element.
Crisp upper naturals are extended naturals

Definition

The **Extended Naturals** $\mathbb{N}^\infty$ is the type of monotone functions $\mathbb{N} \rightarrow \text{Bool}$.

Proposition (Using LEM)

$$\flat \mathbb{N}^\uparrow \simeq \flat \mathbb{N}^\infty$$

And this equivalence restricts to

$$\flat \{\text{Bounded upper naturals}\} \simeq \mathbb{N}$$
The Crisp Countable Axiom of Choice

Axiom (AC$_{\mathbb{N}}$)
Suppose $P :: \mathbb{N} \rightarrow \textbf{Type}$ is a crisp countable family of types. If $f :: (n : \mathbb{N}) \rightarrow \|Pn\|$ crisply, then $\|(n : \mathbb{N}) \rightarrow Pn\|$.

Proposition
Assuming AC$_{\mathbb{N}}$, $\mathbb{N}^{\infty} \simeq \mathbb{N} + \{\infty\}$.
Corollaries

Corollary

- Every crisp type is either infinite or has a natural number cardinality.
- Every crisp polynomial has natural number degree.
- Every crisp vector space has natural number dimension.
- ...
References

