Degrees, Dimensions, and Crispness

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Outline

- The upper naturals.
- The algebra of polynomials, three ways.
- Crisp things have natural number degree / dimension.

The Logic of Space

Space-y-ness of your domains of discourse

$$\Longleftrightarrow$$

Constructiveness of the (native) logic about things in those domains

Logical Connectivity

Definition

A proposition $U : A \to \textbf{Prop}$ is **logically connected** if for all $P : A \to \textbf{Prop}$, if $\forall a.\, Ua \to Pa \vee \neg Pa$, then either $\forall a.\, Ua \to Pa$ or $\forall a.\, Ua \to \neg Pa$.

Lemma

If $U : A \to \textbf{Prop}$ is logically connected and $f : A \to B$, then its image $\mathrm{im}(U) :\equiv \lambda b.\, \exists a.\, f(a) = b \wedge Ua : B \to \textbf{Prop}$ is logically connected.

Lemma

If A has decidable equality (either $a = b$ or $a \neq b$), then a logically connected $U : A \to \textbf{Prop}$ has at most one element.

Degree of a Polynomial

- Suppose R is a ring. Naively, taking the degree of a polynomial should give a map
$$\deg : R[x] \to \mathbb{N}$$
- But suppose that R is logically connected and for $r : R$ consider the polynomial rx.
- Then $\deg(rx) : \mathbb{N}$, so that
$$\lambda r.\, \deg(rx) : R \to \mathbb{N}\,.$$
But R is connected and $\mathbb{N}$ has decidable equality, so this map must be constant (by the lemma).
- Of course, $\deg(x) = 1$ and $\deg(0) = 0$, so this proves $1 = 0$, which is an issue.

Problems with the Naturals

So there's a problem with the naturals – they are too *discrete*. How do we fix this?

To solve this, we need to find another problem with the natural numbers: one from **logic**.

Proposition

The law of excluded middle (LEM) is equivalent to the well-ordering principle (WOP) for $\mathbb{N}$.

Proof.

That the classical naturals satisfy WOP is routine. Let's show that the well-ordering of $\mathbb{N}$ implies LEM.

Given a proposition $P : \mathbf{Prop}$, define $\bar{P} : \mathbb{N} \to \mathbf{Prop}$ by $\bar{P}(n) :\equiv P \vee 1 \leq n$ and note that $\bar{P}(0) = P$.The least number satisfying $\bar{P}$ is 0 or not depending on whether P or $\neg P$; since equality of naturals is decidable, either P or $\neg P$. $\square$

The Upper Naturals

In other words,

> *The naturals are not complete as a* **Prop**-*category.*

So, let's freely complete them! We will replace a natural number $n : \mathbb{N}$ by its upper bounds $\lambda m.\, n \leq m : \mathbb{N} \to \mathbf{Prop}$.

Definition

The **upper naturals** $\mathbb{N}^{\uparrow}$ are the type of upward closed propositions on the naturals. (As a **Prop**-category, this is $\left(\mathbf{Prop}^{\mathbb{N}}\right)^{\mathsf{op}}$)

- We think of an upper natural $N : \mathbb{N}^{\uparrow}$ as a natural "defined by its upper bounds":

  Nn holds if n is an upper bound of N.

- For N, $M : \mathbb{N}^{\uparrow}$, say $N \leq M$ when every upper bound of M is an upper bound of N.

Naturals and Upper Naturals

Definition

The **upper naturals** $\mathbb{N}^{\uparrow}$ are the type of upward closed propositions on the naturals.

Every natural $n : \mathbb{N}$ gives an upper natural $n^{\uparrow} : \mathbb{N}^{\uparrow}$ by the Yoneda embedding:

$$n^{\uparrow}(m) :\equiv n \leq m.$$

and we define $\infty^{\uparrow} :\equiv \lambda\_.\,\mathrm{False}$.

An upper natural $N : \mathbb{N}^{\uparrow}$ is **bounded** if there exists an upper bound $n : \mathbb{N}$ of N (that is, if $\exists n.\, Nn$).

We can take the minimum upper natural satisfying a proposition:

$$\min : (\mathbb{N} \to \mathbf{Prop}) \to \mathbb{N}^{\uparrow}$$

by

$$(\min P)n :\equiv \exists m \leq n.\, Pm$$

Upper Arithmetic

Definition

$$\begin{aligned} \min : (\mathbb{N} \to \textbf{Prop}) &\to \mathbb{N}^{\uparrow} \\ P &\mapsto \lambda n.\, \exists m \leq n.\, Pm \end{aligned}$$

Lemma

For $P : \mathbb{N} \to \textbf{Prop}$, $\min P = n^{\uparrow}$ if and only if n is the least number satisfying P.

We can define the arithmetic operations for upper naturals by Day convolution: (with $N, M : \mathbb{N}^{\uparrow}$)

- $(N + M)n :\equiv \exists a, b : \mathbb{N}\,.\, Na \wedge Mb \wedge (a + b \leq n)$.
- $(N \cdot M)n :\equiv \exists a, b : \mathbb{N}\,.\, Na \wedge Mb \wedge (ab \leq n)$.
- And one can prove the expected identities by the usual Day convolution arguments.

Upper Naturals in Models

- In localic models, $\mathbb{N}^{\uparrow}$ is the sheaf of upper semi-continuous functions valued in $\mathbb{N}$.
- (Hartshorne (1977) Example III.12.7.2) If Y is a Noetherian scheme and $\mathcal{F}$ a coherent sheaf of modules on Y, then

$$y \mapsto \dim_{k(y)}(\mathcal{F}_y \otimes k(y))$$

  is an upper-semicontinuous function $Y \to \mathbb{N}$, and therefore a global section of $\mathbb{N}^{\uparrow} \in \mathbf{Sh}(Y)$.
- For more on the upper naturals in a localic setting, see Section II.5 of Blechschmidt (2017). (There they are called *generalized naturals*)

Cardinality

As an example of what we can define with upper naturals that we couldn't with naturals, consider:

Definition

Define the (finite) **cardinality** of a type as

$$\begin{aligned} \mathbf{Card} : \mathbf{Type} &\to \mathbb{N}^{\uparrow} \\ X &\mapsto \min\left(\lambda n.\ \|[n] \simeq X\|\right) \end{aligned}$$

(or, the Kuratowski cardinality by $X \mapsto \min\left(\lambda n. \exists f : [n] \twoheadrightarrow X\right)$)

Proposition

We have the expected equations:

- $\mathbf{Card}(X + Y) = \mathbf{Card}(X) + \mathbf{Card}(Y)$.
- $\mathbf{Card}(X \times Y) = \mathbf{Card}(X) \cdot \mathbf{Card}(Y)$.
- $\mathbf{Card}(X +_U Y) = \mathbf{Card}(X) + \mathbf{Card}(Y) - \mathbf{Card}(U)$.*

Polynomials, Three Ways

To define the degree of a polynomial, we need to define the algebra of polynomials. In the following, let R be a ring.

Definition

For a type I, the **free R-algebra on I**, $R[x_i \mid i : I]$ is the higher inductive type generated by

- $x : I \to R[x_i \mid i : I]$
- struct : R-algebra structure on $R[x_i \mid i : I]$

Proposition

Let A be an R-algebra and I a type. Then evaluating at $x : I \to R[x_i \mid i : I]$ gives an equivalence

$$(I \to A) \simeq \mathbf{Alg}_R(R[x_i \mid i : I], A).$$

Polynomials, Three Ways

This gives a straightforward definition of $R[x]$ as $R[x_i \mid i : *]$.

But it's not immediately clear how to define the degree of a polynomial using this definition. Let's give another:

Definition

Define $R[x]^s$ to be the type of eventually vanishing sequences in R. That is

$$R[x]^s :\equiv (f : \mathbb{N} \to R) \times \exists n. \forall m > n. f_m = 0.$$

Proposition

Let A be an R-algebra. Then, evaluation at $x : R[x]^s$ gives an equivalence

$$A \simeq \mathbf{Alg}_R(R[x]^s, A).$$

The Degree of a Polynomial

Now we can define

$$\begin{aligned} \deg : R[x]^s &\to \mathbb{N}^{\uparrow} \\ \deg(f)n &\equiv: \forall m > n.\, f_m = 0 \end{aligned}$$

We can prove some basic facts about the degree:

- If $\deg(f) = n^{\uparrow}$, then $f = \sum_{i=0}^{n} f_i x^i$.
- $\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$.
- $\deg(fg) \leq \deg(f) + \deg(g)$.
- What about $\deg(f \circ g) \leq \deg(f) \cdot \deg(g)$?

Horner Normal Form

We note that any polynomial f can be written as

$$f(x) = g(x) \cdot x + f(0)$$

Definition

Let $R[x]^h$ be the higher inductive type given by

- $\mathrm{const} : R \to R[x]^h$,
- $(-) \cdot x + (-) : R[x]^h \times R \to R[x]^h$,
- $\mathrm{eq} : (r : R) \to \mathrm{const}(0) \cdot x + \mathrm{const}(r) = \mathrm{const}(r)$,
- $\mathrm{is-set} : R[x]^h$ is a set.

Proposition

For any R-algebra A, evaluation at $\mathrm{const}(1) \cdot x + \mathrm{const}(0)$ gives an equivalence

$$A \simeq \mathbf{Alg}_R(R[x]^h, A).$$

Induction on ~~Degree~~ Horner Normal Form

Definition

Define the composite $f \circ g$ of two polynomials $f, g : R[x]^h$ by induction on f:

- If $f \equiv \text{const}(r)$, then $f \circ g :\equiv \text{const}(r)$.
- If $f \equiv h \cdot x + \text{const}(r)$, then $f \circ g :\equiv (h \circ g) \cdot g + \text{const}(r)$.
- We check that $(0 \cdot x + r) \circ g = r$, and
- We note we are mapping into a set.

Induction on ~~Degree~~ Horner Normal Form

Proposition

For any polynomials $f, g : R[x]^h$, $\deg(f \circ g) \leq \deg(f) \cdot \deg(g)$.

Proof.

By induction on horner normal form:

$$\begin{aligned}
\deg((f(x)x + r) \circ g) &= \deg((f \circ g)(x) \cdot g(x) + r) \\
&= \deg((f \circ g)(x) \cdot g(x)) \\
&\leq \deg((f \circ g)) + \deg(g) \\
&\leq \deg(f) \cdot \deg(g) + \deg(g) && \text{by hypothesis} \\
&= (\deg(f) + 1^{\uparrow}) \cdot \deg(g) \\
&= \deg(f(x) \cdot x + r) \cdot \deg(g)
\end{aligned}$$

$\square$

Slogan: Instead of inducting on degree, induct on the polynomial!

Dimension

Definition

We define the dimension of a vector space V over a field k by

$$(\dim V)n :\equiv \min(\lambda n.\ \|k^n \cong V\|)$$

It is the minimum n such that V has an n-element basis

Proposition

Let $f : k[x]$. Then $\deg(f) = \dim(k[x]/(f))$.

Catching up on Crispness

- Recall that Shulman's cohesive homotopy type theory uses **crisp** variables to keep track of discontinuous dependency. A term is crisp if all the free variables in it are crisp.
- Crisp variables must have crisp type, and only crisp terms can be substituted for crisp variables.
- So, $x :: X$ – a crisp point of X – is a general *discontinuous* element of X.

Axiom (LEM)

If P :: **Prop** is a crisp proposition, then either P or $\neg P$ holds.

Discontinuously, every proposition is either true or false.

Crisp upper naturals are extended naturals

- If X is a crisp type, then $\flat X$ can be thought of as the type of crisp points of X.

Definition

The **Extended Naturals** $\mathbb{N}^\infty$ is the type of monotone functions $\mathbb{N} \to \mathrm{Bool}$. Equivalently, it is the type of upwards-closed *decidable* propositions on the naturals.

Proposition

- The extended naturals embed into the upper naturals, preserving the naturals.
- The bounded extended naturals are equivalent to the naturals. Every decidable, inhabited subset of $\mathbb{N}$ has a least element.

Crisp upper naturals are extended naturals

Definition

The **Extended Naturals** $\mathbb{N}^\infty$ is the type of monotone functions $\mathbb{N} \to \mathsf{Bool}$.

Proposition (Using LEM)

$$\flat\, \mathbb{N}^{\uparrow} \simeq \flat\, \mathbb{N}^{\infty}$$

And this equivalence restricts to

$$\flat\{\text{Bounded upper naturals}\} \simeq \mathbb{N}$$

The Crisp Countable Axiom of Choice

Axiom ($\mathrm{AC}_{\mathbb{N}}$)

Suppose $P :: \mathbb{N} \to \textbf{Type}$ is a crisp countable family of types. If $f :: (n : \mathbb{N}) \to \|Pn\|$ crisply, then $\|(n : \mathbb{N}) \to Pn\|$.

Proposition

Assuming $\mathrm{AC}_{\mathbb{N}}$, $\flat\, \mathbb{N}^{\infty} \simeq \mathbb{N} + \{\infty\}$.

Corollaries

Corollary

- Every crisp type is either infinite or has a natural number cardinality.
- Every crisp polynomial has natural number degree.
- Every crisp vector space has natural number dimension.
- ...

References

Ingo Blechschmidt. Using the internal language of toposes in algebraic geometry. *Phd Thesis*, 2017.

Henri Lombardi and Claude Quitté. Commutative algebra: Constructive methods. Finite projective modules. *arXiv e-prints*, art. arXiv:1605.04832, May 2016.

Michael Shulman. Brouwer's fixed-point theorem in real-cohesive homotopy type theory. *arXiv e-prints*, art. arXiv:1509.07584, Sep 2015.