A Foundation for Synthetic Algebraic Geometry

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Abstract

The following is work in progress on a development of algebraic geometry, internal to the Zariski topos, building on the work of Kock and Blechschmidt ([Koc06][I.12], [Ble17]). The Zariski topos consists of sheaves on the site opposite to the category of finitely presented algebras over a fixed ring, with the Zariski topology, i.e. generating covers are given by localization maps $A \to A_{f_1}$ for finitely many elements $f_1, \dots, f_n$ that generate the ideal $(1) = A \subseteq A$.

Since we use HoTT, we need a higher version of these sheaves. One innovation we present, is the use of higher types to define and reason about cohomology. Actually computing cohomology groups, seems to need a principle along the lines of our "Zariski local choice" axiom, which we justify using a cubical model of homotopy type theory.

Contents

Introduction

Algebraic geometry is the study of solutions of non-linear equations using methods from geometry. Most prominently, algebraic geometry was essential in the proof of Fermat's last theorem by Wiles and Taylor. The central geometric objects in algebraic geometry are called *schemes*. Their basic building blocks are called *affine schemes*, where, informally, an affine scheme corresponds to a solution sets of polynomial equations. While this correspondence is clearly visible in the functorial approach to algebraic geometry and our synthetic approach, it is somewhat obfuscated in the most commonly used, topological appraoch.

In recent years, formalization of the intricate notion of affine schemes received some attention as a benchmark problem – this is, however, *not* a problem addressed by this work. Instead, we use a synthetic approach to algebraic geometry, very much alike to that of synthetic differential geometry. This means, while a scheme in classical algebraic geometry is a complicated compound datum, we work in a setting where schemes are types, with an additional property that can be defined within our synthetic theory.

In our work in progress [CCH23], following ideas of Ingo Blechschmidt and Anders Kock ([Ble17], [Koc06][I.12]), we use a base ring R, which is local and satisfies an axiom reminiscent of the Kock-Lawvere axiom. A more general axiom, is called *synthetic quasi coherence (SQC)* by Blechschmidt and a version quatifying over external algbras is called the *comprehensive axiom*[1] by Kock. The exact concise form of the SQC axiom we use, was noted by David Jaz Myers in 2018 and communicated to the first author.

Before we state the SQC axiom, let us take a step back and look at the basic objects of study in

[1]In [Koc06][I.12], Kock's "axiom 2_k" could equivalently be Theorem 12.2, which is exactly our synthetic quasi coherence axiom, except that it only quantifies over external algebras.

algebraic geometry, solutions of polynomial equations. Given a system of polynomial equations

$$\begin{aligned} p_1(X_1,\dots,X_n) &= 0,\\ &\vdots\\ p_m(X_1,\dots,X_n) &= 0, \end{aligned}$$

the solution set $\{x : R^n \mid \forall i.\ p_i(x_1,\dots,x_n) = 0\}$ is in canonical bijection to the set of R-algebra homomorphisms

$$\mathrm{Hom}_R(R[X_1,\dots,X_n]/(p_1,\dots,p_m), R)$$

by identifying a solution $(x_1,\dots,x_n)$ with the homomorphism that maps each X_i to x_i. Conversely, for any R-algebra A, which is merely of the form $R[X_1,\dots,X_n]/(p_1,\dots,p_m)$, we define the *spectrum* of A to be

$$\operatorname{Spec} A :\equiv \mathrm{Hom}_R(A, R).$$

In contrast to classical, non-synthetic algebraic geometry, where this set needs to be equipped with additional structure, we postulate axioms that will ensure that $\operatorname{Spec} A$ has the expected geometric properties. Namely, SQC is the statement that, for all finitely presented R-algebras A, the canonical map

$$\begin{aligned} A &\xrightarrow{\sim} (\operatorname{Spec} A \to R)\\ a &\mapsto (\varphi \mapsto \varphi(a)) \end{aligned}$$

is an equivalence. A prime example of a spectrum is $\mathbb{A}^1 :\equiv \operatorname{Spec} R[X]$, which turns out to be the underlying set of R. With the SQC axiom, *any* function $f : \mathbb{A}^1 \to \mathbb{A}^1$ is given as a polynomial with coefficients in R. In fact, all functions between affine schemes are given by polynomials. Furthermore, for any affine scheme $\operatorname{Spec} A$, the axiom ensures that the algebra A can be reconstructed as the algebra of functions $\operatorname{Spec} A \to R$, therefore establishing a duality between affine schemes and algebras.

The Kock-Lawvere axiom used in synthetic differential geometry, might be stated as the SQC axiom restricted to (external) *Weil-algebras*, whose spectra correspond to pointed infinitesimal spaces. These spaces can be used in both, synthetic differential and algebraic geometry, in very much the same way.

In the accompanying formalization [CH] of some basic results, we use a setup which was already proposed by David Jaz Myers in a conference talk ([Mye19b; Mye19a]). On top of Myers' ideas, we were able to define schemes, develop some topological properties of schemes, and construct projective space.

An important, not yet formalized result is the construction of cohomology groups. This is where the *homotopy* type theory really comes to bear – instead of the hopeless adaption of classical, non-constructive definitions of cohomology, we make use of higher types, for example the k-th Eilenberg-MacLane space $K(R,k)$ of the group $(R,+)$. As an analogue of classical cohomology with values in the structure sheaf, we then define cohomology with coefficients in the base ring as:

$$H^k(X,R) :\equiv \|X \to K(R,k)\|_0.$$

This definition is very convenient for proving abstract properties of cohomology. For concrete calculations we make use of another axiom, which we call *Zariski-local choice*. While this axiom was conceived of for exactly these kind of calculations, it turned out to settle numerous questions with no apparent connection to cohomology. One example is the equivalence of two notions of *open subspace*. A pointwise definition of openness was suggested to us by Ingo Blechschmidt and is very convenient to work with. However, classically, basic open subsets of an affine scheme are given by functions on the scheme and the corresponding open is morally the collection of points where the function does not vanish. With Zariski-local choice, we were able to show that these notions of openness agree in our setup.

Apart from SQC, locality of the base ring R and Zariski-local choice, we only use homotopy type theory, including univalent universes, truncations and some very basic higher inductive types. Roughly, Zariski-local choice states, that any surjection into an affine scheme merely has sections on a *Zariski*-cover. The latter, internal, notion of cover corresponds quite directly to the covers in the site of the *Zariski topos*, which we use to construct a model of homotopy type theory with our axioms.

More precisely, we can use the *Zariski topos* over any base ring. Toposes built using other Grothendieck topologies, like for example the étale topology, are not compatible with Zariski-local choice. We did not explore whether an analogous setup can be used for derived algebraic geometry [2] – meaning that the

[2] Here, the word "derived" refers to the rings the algebraic geometry is built up from – instead of the 0-truncated rings we use, "derived" algebraic geometry would use simplicial or spectral rings. Sometimes, "derived" refers to homotopy types appearing in "the other direction", namely as the values of the sheaves that used. In that direction, our theory is already derived, since we use homotopy type theory. Practically that means that we expect no problems when expanding our theory of synthetic schemes to what classic algebraic geometers call "stacks".

0-truncated rings we used are replaced by higher rings. This is only because for a derived approach, would have to work with higher monoids, which is currently infeasible – we are not aware of any obstructions for, say, an SQC axiom holding in derived algebraic geometry.

In total, the scope of our theory so far, includes quasi-compact, quasi-separated schemes of finite type over an arbitrary ring. These are all finiteness assumptions, that were chosen for convenience and include examples like closed subspaces of projective space, which we want to study in future work, as example applications. So far, we know that basic internal constructions, like affine schemes, correspond to the correct classical external constructions. This can be expanded using our model, which is of course also important to ensure the consistency of our setup.

Formalization

There is a related formalization project, which, at the time of writing, contains the construction of projective n-space $\mathbb{P}^n$ as a scheme. The code may be found here:

https://github.com/felixwellen/synthetic-geometry

It makes extensive use of the algebra part of the cubical-agda library:

https://github.com/agda/cubical

– which contains many contributions, in particular, on finitely presented algebras and related concepts, which where made in the scope of that project.

Acknowlegements

We use work from Ingo Blechschmidt's phd-thesis, section 18 as a basis. This includes in particular the synthetic quasi-coherence axiom and the assumption that the base ring is local. David Jaz Myers had the idea to use Blechschmidt's ideas in homotopy type theory and presented his ideas 2019 at the workshop "Geometry in Modal Homotopy Type Theory" in Pittsburgh. Myers ideas include the algebra-setup we used in our formalization.

In December 2022, there was a mini-workshop in Augsburg, which helped with the development of this work. We thank Jonas Höfer for carefully reading a draft.

1 Preliminaries

1.1 Subtypes and Logic

We use the notation $\exists_{x:X} P(x) :\equiv \| \sum_{x:X} P(x)\|$. We use $+$ for the coproduct of types and for types A, B we write

$$A \vee B :\equiv \|A + B\|.$$

We will use subtypes extensively.

Definition 1.1.1 Let X be a type. A *subtype* of X is a function $U : X \to \mathrm{Prop}$ to the type of propositions. We write $U \subseteq X$ to indicate that U is as above. If X is a set, a subtype may be called *subset* for emphasis. For subtypes $A, B \subseteq X$, we write $A \subseteq B$ as a shorthand for pointwise implication.

We will freely switch between subtypes $U : X \to \mathrm{Prop}$ and the corresponding embeddings

$$\textstyle\sum_{x:X} U(x) \hookrightarrow X \ .$$

In particular, if we write $x : U$ for a subtype $U : X \to \mathrm{Prop}$, we mean that $x : \sum_{x:X} U(x)$ – but we might silently project x to X.

Definition 1.1.2 Let I and X be types and $U_i : X \to \mathrm{Prop}$ a subtype for any $i : I$.

(a) The *union* $\bigcup_{i:I} U_i$ is the subtype $(x : X) \mapsto \exists_{i:I} U_i(x)$.

(b) The *intersection* $\bigcap_{i:I} U_i$ is the subtype $(x : X) \mapsto \prod_{i:I} U_i(x)$.

We will use common notation for finite unions and intersections. The following formula hold:

Lemma 1.1.3 Let I, X be types, $U_i : X \to \mathrm{Prop}$ a subtype for any $i : I$ and V, W subtypes of X.

(a) Any subtype $P : V \to \mathrm{Prop}$ is a subtype of X given by $(x : X) \mapsto \sum_{x:V} P(x)$.

(b) $V \cap \bigcup_{i:I} U_i = \bigcup (V \cap U_i)$.

(c) If $\bigcup_{i:I} U_i = X$ we have $V = \bigcup_{i:I} U_i \cap V$.

(d) If $\bigcup_{i:I} U_i = \emptyset$, then $U_i = \emptyset$ for all $i : I$.

Definition 1.1.4 Let X be a type.

(a) $\emptyset :\equiv (x : X) \mapsto \emptyset$.

(b) For $U \subseteq X$, let $\neg U :\equiv (x : X) \mapsto \neg U(x)$.

(c) For $U \subseteq X$, let $\neg\neg U :\equiv (x : X) \mapsto \neg\neg U(x)$.

Lemma 1.1.5 $U = \emptyset$ if and only if $\neg\left(\exists_{x:X} U(x)\right)$.

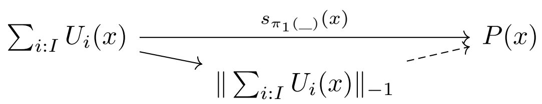
1.2 Homotopy type theory

Our truncation levels start at -2, so (-2)-types are contractible, (-1)-types are propositions and 0-types are sets.

Definition 1.2.1 Let X and I be types. A family of propositions $U_i : X \to \mathrm{Prop}$ *covers* X, if for all $x : X$, there merely is a $i : I$ such that $U_i(x)$.

Lemma 1.2.2 Let X and I be types. For propositions $(U_i : X \to \mathrm{Prop})_{i:I}$ that cover X and $P : X \to$ 0-Type, we have the following glueing property:
If for each $i : I$ there is a dependent function $s_i : (x : U_i) \to P(x)$ together with proofs of equality on intersections $p_{ij} : (x : U_i \cap U_j) \to (s_i(x) = s_j(x))$, then there is a globally defined dependent function $s : (x : X) \to P(x)$, such that for all $x : X$ and $i : I$ we have $U_i(x) \to s(x) = s_i(x)$

Proof We define s pointwise. Let $x : X$. Using a Lemma of Kraus and the p_{ij}, we get a factorization



– which defines a unique value $s(x) : P(x)$. □

Similarly we can prove.

Lemma 1.2.3 Let X and I be types. For propositions $(U_i : X \to \mathrm{Prop})_{i:I}$ that cover X and $P : X \to$ 1-Type, we have the following glueing property:
If for each $i : I$ there is a dependent function $s_i : (x : U_i) \to P(x)$ together with proofs of equality on intersections $p_{ij} : (x : U_i \cap U_j) \to (s_i(x) = s_j(x))$ satsfying the cocycle condition $p_{ij} \cdot p_{jk} = p_{ik}$. then there is a globally defined dependent function $s : (x : X) \to P(x)$, such that for all $x : X$ and $i : I$ we have $p_i : U_i(x) \to s(x) = s_i(x)$ such that $p_i \cdot p_{ij} = p_j$.

This can be generalized to k-Type for each *external* k.

The condition for 0-Type can be seen as an internal version of the usual patching *sheaf* condition. The condition for 1-Type is then the internal version of the usual patching 1-*stack* condition.

1.3 Algebra

Definition 1.3.1 A commutative ring R is *local* if $1 \neq 0$ in R and if for all $x, y : R$ such that $x + y$ is invertible, x is invertible or y is invertible.

Definition 1.3.2 Let R be a commutative ring. A *finitely presented* R-algebra is an R-algebra A, such that there merely are natural numbers n, m and polynomials $f_1, \ldots, f_m : R[X_1, \ldots, X_n]$ and an equivalence of R-algebras $A \simeq R[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$.

Definition 1.3.3 Let A be a commutative ring. An element $r : A$ is *regular*, if the multiplication map $r \cdot \_ : A \to A$ is injective.

Lemma 1.3.4 Let A be a commutative ring.

(a) All units of A are regular.

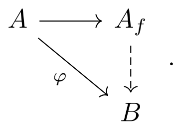
(b) If f and g are regular, their product fg is regular.

Example 1.3.5 The monomials $X^k : A[X]$ are regular.

Lemma 1.3.6 Let $f : A[X]$ be a polynomial and $a : A$ an element such that $f(a) : A$ is regular. Then f is regular as an element of $A[X]$.

Proof TODO □

Definition 1.3.7 Let A be a ring and $f : A$. Then A_f denotes the *localization* of A at f, i.e. a ring A_f together with a homomorphism $A \to A_f$, such that for all homomorphisms $\varphi : A \to B$ such that $\varphi(f)$ is invertible, there is a unique homomorphism as indicated in the diagram:



For $a : A$, we denote the image of a in A_f as $\frac{a}{1}$ and the inverse of f as $\frac{1}{f}$.

Lemma 1.3.8 Let A be a commutative ring and $f_1, \dots, f_n : A$. For finitely generated ideals $I_i \subseteq A_{f_i}$, such that $A_{f_i f_j} \cdot I_i = A_{f_i f_j} \cdot I_j$ for all i, j, there is a finitely generated ideal $I \subseteq A$, such that $A_{f_i} \cdot I = I_i$ for all i.

Proof Choose generators

$$\frac{g_{i1}}{1}, \dots, \frac{g_{ik_i}}{1}$$

for each I_i. These generators will still generate I_i, if we multiply any of them with any power of the unit $\frac{f_i}{1}$. Now

$$A_{f_i f_j} \cdot I_i \subseteq A_{f_i f_j} \cdot I_j$$

means that for any g_{ik}, we have a relation

$$(f_i f_j)^l g_{ik} = \sum_l h_l g_{jl}$$

for some power l and coefficients $h_l : A$. This means, that $f_i^l g_{ik}$ is contained in I_j. Multiplying $f_i^l g_{ik}$ with further powers of f_i or multiplying g_{jl} with powers of f_j does not change that. So we can repeat this for all i and k to arrive at elements $\tilde{g_{ik}} : A$, which generate an ideal $I \subseteq A$ with the desired properties. □

The following definition also appears as [Ble17][Definition 18.5] and a version restricted to external finitely presented algebras was already used by Anders Kock in [Koc06][I.12]:

Definition 1.3.9 The *(synthetic) spectrum* of a finitely presented R-algebra A is the set of R-algebra homomorphisms from A to R:

$$\operatorname{Spec} A :\equiv \operatorname{Hom}_{R\text{-Alg}}(A, R)$$

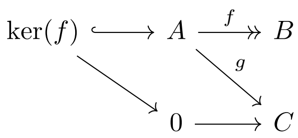
Definition 1.3.10 Let A be a finitely presented R-algebra. For $f : A$, the *standard open subset* given by f, is the subtype

$$D(f) :\equiv (x : \operatorname{Spec} A) \mapsto (x(f) \text{ is invertible}).$$

Definition 1.3.11 Ab denotes the type of abelian groups.

Lemma 1.3.12 Let A, B : Ab and $f : A \to B$ be a homomorphism of abelian groups. Then f is surjective, if and only if, it is a cokernel.

Proof A cokernel is a set-quotient by an effective relation, so the projection map is surjective. On the other hand, if f is surjective and we are in the situation:



then we can construct a map $\varphi : B \to C$ as follows. For $x : B$, we define the type of possible values $\varphi(x)$ in C as

$$\sum_{z:C} \exists_{y:A}(f(y) = x)$$

which is a proposition by algebraic calculation. By surjectivity of f, this type is inhabited and therefore contractible. So we can define $\varphi(x)$ as its center of contraction. □

2 Axioms

2.1 Statement of the axioms

We always assume there is a commutative ring R. Sometimes we will assume R has additional properties, or, more generally, axioms hold that involve R. We will always mention which of these axioms are needed to prove each statement, by listing the shorthands introduced in the axioms below.

Axiom (Loc)
R is a local ring.

Axiom (SQC)
For any finitely presented R-algebra A, the homomorphism

$$a \mapsto (\varphi \mapsto \varphi(a)) : A \to (\operatorname{Spec} A \to R)$$

is an isomorphism of R-algebras.

Axiom (Z-choice)
Let A be a finitely presented R-algebra and let $B : \operatorname{Spec} A \to \mathcal{U}$ be a family of inhabited types. Then there merely exists a finite list of coprime elements $f_1, \ldots, f_n \in A$ together with dependent functions $s_i : \Pi_{x:D(f_i)} B(x)$. As a formula:

$$(\Pi_{x:\operatorname{Spec} A} \|B(x)\|) \to \|\Sigma_{n:\mathbb{N}} \Sigma_{f_1,\ldots,f_n:A}((f_1, \ldots, f_n) = (1)) \times \Pi_i \Pi_{x:D(f_i)} B(x)\|.$$

2.2 First consequences

Proposition 2.2.1 (using SQC) For all finitely generated R-algebras A and B we have

$$(\operatorname{Spec} B \to \operatorname{Spec} A) = \operatorname{Hom}_{R\text{-Alg}}(A, B)$$

– where the equality is induced by exponentiation with R.

An important consequence, which may be called *weak nullstellensatz*:

Proposition 2.2.2 (using Loc, SQC) If A is a finitely generated R-algebra, $\operatorname{Spec} A = \emptyset$, if and only if $A = 0$.

Proof If $\operatorname{Spec} A = \emptyset$ then $A = R^{\operatorname{Spec} A} = R^{\emptyset} = 0$ by (SQC). If $A = 0$ then there are no homomorphisms $A \to R$ since $1 \neq 0$ in R by (Loc). □

The following are originally proven in [Ble17][Section 18]:

Proposition 2.2.3 (using SQC, Loc) (a) An element $x : R$ is nilpotent, if and only if $\neg\neg(x = 0)$.

(b) An element $x : R$ is invertible, if and only if $x \neq 0$.

(c) A vector $x : R^n$ is non-zero, if and only if one of its entries is invertible.

3 Affine schemes

3.1 Affine-open subtypes

We only talk about affine schemes of finite type, i.e. schemes of the form $\operatorname{Spec} A$ (definition 1.3.9), where A is a finitely presented algebra.

Definition 3.1.1 A type X is *(qc-)affine*, if there is a finitely presented R-algebra A, such that $X = \operatorname{Spec} A$.

Proposition 3.1.2 Let X be a type. The type of all finitely presented R-algebras A, such that $X = \operatorname{Spec} A$, is a proposition.

When we write "$\operatorname{Spec} A$" we implicitly assume A is a finitely presented R-algebra. Recall from definition 1.3.10 that the standard open subset $D(f) \subseteq \operatorname{Spec} A$ is given by $D(f)(x) :\equiv \operatorname{inv}(f(x))$.

Example 3.1.3 (using Loc, SQC) For $a_1, \ldots, a_n : R$, we have

$$D((X - a_1) \ldots (X - a_n)) = \mathbb{A}^1 \setminus \{a_1, \ldots, a_n\}.$$

Indeed, for any $x : \mathbb{A}^1$, $((X - a_1) \ldots (X - a_n))(x)$ is invertible if and only if $x - a_i$ is invertible for all i. But by proposition 2.2.3 this means $x \neq a_i$ for all i.

Definition 3.1.4 Let $X = \operatorname{Spec} A$. A subtype $U : X \to \mathrm{Prop}$ is called *affine-open*, if one of the following logically equivalent statements holds:

(i) U is the union of finitely many affine standard opens.

(ii) There are $f_1, \ldots, f_n : A$ such that
$$U(x) \Leftrightarrow \exists_i f_i(x) \neq 0$$

We sometimes write $D(f_1, \ldots, f_n) :\equiv D(f_1) \cup \cdots \cup D(f_n)$ for a finite union of standard opens. Note that in general, affine-open subtypes do not need to be affine – this is why we use the dash "-".

We will introduce a more general definition of open subtype in definition 4.2.1 and show in theorem 4.2.7, that the two notions agree on affine schemes.

Proposition 3.1.5 Let $X = \operatorname{Spec} A$ and $f : A$. Then $D(f) = \operatorname{Spec} A[f^{-1}]$.

Proof

$$D(f) = \sum_{x:X} D(f)(x) = \sum_{x:\operatorname{Spec} A} \operatorname{inv}(f(x)) = \sum_{x:\operatorname{Hom}_{R\text{-Alg}}(A,R)} \operatorname{inv}(x(f)) = \operatorname{Hom}_{R\text{-Alg}}(A[f^{-1}], R) = \operatorname{Spec} A[f^{-1}]$$

□

Affine-openness is transitive in the following sense:

Lemma 3.1.6 Let $X = \operatorname{Spec} A$ and $D(f) \subseteq X$ be a standard open. Any affine-open subtype U of $D(f)$ is also affine-open in X.

Proof It is enough to show the statement for $U = D(g)$, $g : A_f$. Then

$$g = \frac{h}{f^k}.$$

Now $D(hf)$ is an affine-open in X, that coincides with U:
Let $x : X$, then $(hf)(x)$ is invertible, if and only if both $h(x)$ and $f(x)$ are invertible. The latter means $x : D(f)$, so we can interpret x as a homorphism from A_f to R. Then $x : D(g)$ means $x(g)$ is invertible, which is equivalent to $x(h)$ being invertible, since $x(f)^k$ is invertible anyway. □

Lemma 3.1.7 (using Loc, SQC) Let $X = \operatorname{Spec} A$ be an affine scheme and $D(f) \subseteq X$ a standard open, then $D(f) = \emptyset$, if and only if, f is nilpotent.

Proof Since $D(f) = \operatorname{Spec} A_f$, by proposition 2.2.2, we know $D(f) = \emptyset$, if and only if, $A_f = 0$. The latter is equivalent to f being nilpotent. □

More generally, the Zariski-lattice consisting of the radicals of finitely generated ideals of a finitely presented R-algebra A, coincides with the lattice of open subtypes. This means, that internal to the Zariski-topos, it is not neccessary to consider the full Zariski-lattice for a constructive treatment of schemes.

Lemma 3.1.8 (using SQC) Let A be a finitely presented R-algebra and let $f, g_1, \ldots, g_n \in A$. Then we have $D(f) \subseteq D(g_1, \ldots, g_n)$ as subsets of $\operatorname{Spec} A$ if and only if $f \in \sqrt{(g_1, \ldots, g_n)}$.

Proof Since $D(g_1, \ldots, g_n) = \{ x \in \operatorname{Spec} A \mid x \notin V(g_1, \ldots, g_n) \}$, the inclusion $D(f) \subseteq D(g_1, \ldots, g_n)$ can also be written as $D(f) \cap V(g_1, \ldots, g_n) = \varnothing$, that is, $\operatorname{Spec}((A/(g_1, \ldots, g_n))[f^{-1}]) = \varnothing$. By (SQC) this means that the finitely presented R-algebra $(A/(g_1, \ldots, g_n))[f^{-1}]$ is zero. And this is the case if and only if f is nilpotent in $A/(g_1, \ldots, g_n)$, that is, if $f \in \sqrt{(g_1, \ldots, g_n)}$, as stated. □

In particular, we have $\operatorname{Spec} A = \bigcup_{i=1}^n D(f_i)$ if and only if $(f_1, \ldots, f_n) = (1)$.

3.2 Pullbacks of affine schemes

Lemma 3.2.1 The product of two affine schemes is again an affine scheme, namely $\operatorname{Spec} A \times \operatorname{Spec} B = \operatorname{Spec}(A \otimes_R B)$.

Proof By the universal property of the tensor product $A \otimes_R B$. □

More generally we have:

Lemma 3.2.2 (using SQC) Let $X = \operatorname{Spec} A, Y = \operatorname{Spec} B$ and $Z = \operatorname{Spec} C$ be affine schemes with maps $f : X \to Z$, $g : Y \to Z$. Then the pullback of this diagram is an affine scheme given by $\operatorname{Spec}(A \otimes_C B)$.

Proof The maps $f : X \to Z$, $g : Y \to Z$ are induced by R-algebra homomorphisms $f^* : A \to R$ and $g^* : B \to R$. Let

$$(h, k, p) : \operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B$$

with $p : h \circ f^* = k \circ g^*$. This defines a R-cocone on the diagram

$$A \xleftarrow{f^*} C \xrightarrow{g^*} B$$

Since $A \otimes_C B$ is a pushout in R-algebras, there is a unique R-algebra homomorphism $A \otimes_C B \to R$ corresponding to (h, k, p). □

3.3 Boundedness of functions to $\mathbb{N}$

While the axiom SQC describes functions on an affine scheme with values in R, we can generalize it to functions taking values in another finitely presented R-algebra, as follows.

Lemma 3.3.1 (using SQC) For finitely presented R-algebras A and B, the function

$$\begin{aligned} A \otimes B &\xrightarrow{\sim} (\operatorname{Spec} A \to B) \\ c &\mapsto (\varphi \mapsto (\varphi \otimes B)(c)) \end{aligned}$$

is a bijection.

Proof We recall $\operatorname{Spec}(A \otimes B) = \operatorname{Spec} A \times \operatorname{Spec} B$ from lemma 3.2.1 and calculate as follows.

$$\begin{array}{ccccc} A \otimes B = & (\operatorname{Spec}(A \otimes B) \to R) = & (\operatorname{Spec} A \times \operatorname{Spec} B \to R) = & (\operatorname{Spec} A \to (\operatorname{Spec} B \to R)) = & (\operatorname{Spec} A \to B) \\ c \mapsto & (\chi \mapsto \chi(c)) \mapsto & ((\varphi, \psi) \mapsto (\varphi \otimes \psi)(c)) \mapsto & (\varphi \mapsto (\psi \mapsto (\varphi \otimes \psi)(c))) \mapsto & (\varphi \mapsto (\varphi \otimes B)(c)) \end{array}$$

The last step is induced by the identification $B = (\operatorname{Spec} B \to R)$, $b \mapsto (\psi \mapsto \psi(b))$, and we use the fact that $\psi \circ (\varphi \otimes B) = \varphi \otimes \psi$. □

Lemma 3.3.2 (using SQC) Let A be a finitely presented R-algebra and let $s : \operatorname{Spec} A \to (\mathbb{N} \to R)$ be a family of sequences, each of which eventually vanishes:

$$\prod_{x : \operatorname{Spec} A} \| \sum_{N : \mathbb{N}} \prod_{n \geq N} s(x)(n) = 0 \|$$

Then there merely exists one number $N : \mathbb{N}$ such that $s(x)(n) = 0$ for all $x : \operatorname{Spec} A$ and all $n \geq N$.

Proof The set of eventually vanishing sequences $\mathbb{N} \to R$ is in bijection with the set $R[X]$ of polynomials, by taking the entries of a sequence as the coefficients of a polynomial. So the family of sequences s is equivalently a family of polynomials $s : \operatorname{Spec} A \to R[X]$. Now we apply lemma 3.3.1 with $B = R[X]$ to see that such a family corresponds to a polynomial $p : A[X]$. Note that for a point $x : \operatorname{Spec} A$, the homomorphism

$$x \otimes R[X] : A[X] = A \otimes R[X] \to R \otimes R[X] = R[X]$$

simply applies the homomorphism x to every coefficient of a polynomial, so we have $(s(x))_n = x(p_n)$. This concludes our argument, because the coefficients of p, just like any polynomial, form an eventually vanishing sequence. □

Theorem 3.3.3 (using (Loc), (SQC))
Let A be a finitely presented R-algebra. Then every function $f : \operatorname{Spec} A \to \mathbb{N}$ is bounded:

$$\Pi_{f:\operatorname{Spec} A \to \mathbb{N}} \| \Sigma_{N:\mathbb{N}} \Pi_{x:\operatorname{Spec} A} f(x) \leq N \|.$$

Proof Given a function $f : \operatorname{Spec} A \to \mathbb{N}$, we construct the family $s : \operatorname{Spec} A \to (\mathbb{N} \to R)$ of eventually vanishing sequences given by

$$s(x)(n) :\equiv \begin{cases} 1 & \text{if } n < f(x) \\ 0 & \text{else.} \end{cases}$$

Since $0 \neq 1 : R$ by Loc, we in fact have $s(x)(n) = 0$ if and only if $n \geq f(x)$. Then the claim follows from lemma 3.3.2. □

If we also assume the axiom Z-choice, we can formulate the following simultaneous strengthening of lemma 3.3.2 and theorem 3.3.3.

Proposition 3.3.4 (using Loc, SQC, Z-choice) Let A be a finitely presented R-algebra. Let $P : \operatorname{Spec} A \to (\mathbb{N} \to \mathrm{Prop})$ be a family of upwards closed, merely inhabited subsets of $\mathbb{N}$. Then the set

$$\bigcap_{x:\operatorname{Spec} A} P(x) \subseteq \mathbb{N}$$

is merely inhabited.

Proof By Z-choice, there merely exists a cover $\operatorname{Spec} A = \bigcup_{i=1}^n D(f_i)$ and functions $p_i : D(f_i) \to \mathbb{N}$ such that $p_i(x) \in P(x)$ for all $x : D(f_i)$. By theorem 3.3.3, every $p_i : D(f_i) = \operatorname{Spec} A[f_i^{-1}] \to \mathbb{N}$ is merely bounded by some $N_i : \mathbb{N}$, and then $\max(N_1, \dots, N_n) \in P(x)$ for all $x : \operatorname{Spec} A$. □

3.4 Properties of the ring R

We adopt the following definition from [LQ15, Section IV.8].

Definition 3.4.1 A ring A is *zero-dimensional* if for all $x : A$ there exists $a : A$ and $k : \mathbb{N}$ such that $x^k = ax^{k+1}$.

Lemma 3.4.2 (using Loc, SQC, Z-choice) The ring R is not zero-dimensional.

Proof Assume that R is zero-dimensional, so for every $f : R$ there merely is some $k : \mathbb{N}$ with $f^k \in (f^{k+1})$. We note that $R = \mathbb{A}^1$ is an affine scheme and that if $f^k \in (f^{k+1})$, then we also have $f^{k'} \in (f^{k'+1})$ for every $k' \geq k$. This means that we can apply proposition 3.3.4 and merely obtain a number $K : \mathbb{N}$ such that $f^K \in (f^{K+1})$ for all $f : R$. In particular, $f^{K+1} = 0$ implies $f^K = 0$, so the canonical map $\operatorname{Spec} R[X]/(X^K) \to \operatorname{Spec} R[X]/(X^{K+1})$ is a bijection. But this is a contradiction, since the homomorphism $R[X]/(X^{K+1}) \to R[X]/(X^K)$ is not an isomorphism. □

The following lemma, which is a variant of [Ble17][Proposition 18.32], shows that R is in a weak sense algebraically closed.

Lemma 3.4.3 (using Loc, SQC) Let $f : R[X]$ be a polynomial. Then it is not not the case that: either $f = 0$ or $f = \alpha \cdot (X - a_1)^{e_1} \dots (X - a_n)^{e_n}$ for some $\alpha : R^\times$, $e_i \geq 1$ and pairwise distinct $a_i : R$.

Proof Let $f : R[X]$ be given. Since our goal is a proposition, we can assume we have a bound n on the degree of f, so

$$f = \sum_{i=0}^{n} c_i X^i.$$

Since our goal is even double-negation stable, we can assume $c_n = 0 \vee c_n \neq 0$ and by induction $f = 0$ (in which case we are done) or $c_n \neq 0$. If $n = 0$ we are done, setting $\alpha :\equiv c_0$. Otherwise, f is not invertible (using $0 \neq 1$ by (Loc)), so $R[X]/(f) \neq 0$, which by (SQC) means that $\mathrm{Spec}(R[X]/(f)) = \{x : R \mid f(x) = 0\}$ is not empty. Using the double-negation stability of our goal again, we can assume $f(a) = 0$ for some $a : R$ and factor $f = (X - a_1)f_{n-1}$. By induction, we get $f = \alpha \cdot (X - a_1) \dots (X - a_n)$. Finally, we decide each of the finitely many propositions $a_i = a_j$, which we can assume is possible because our goal is still double-negation stable, to get the desired form $f = \alpha \cdot (X - \widetilde{a}_1)^{e_1} \dots (X - \widetilde{a}_n)^{e_n}$ with distinct $\widetilde{a}_i$. □

Example 3.4.4 (using Loc, SQC, Z-choice) It is not the case that every monic polynomial $f : R[X]$ with $\deg f \geq 1$ has a root. More specifically, if $U \subseteq \mathbb{A}^1$ is an open subset with the property that the polynomial $X^2 - a : R[X]$ merely has a root for every $a : U$, then $U = \emptyset$.

Proof Let $U \subseteq \mathbb{A}^1$ be as in the statement. Since we want to show $U = \emptyset$, we can assume a given element $a_0 : U$ and now have to derive a contradiction. By Z-choice, there exists in particular a standard open $D(f) \subseteq \mathbb{A}^1$ with $a_0 \in D(f)$ and a function $g : D(f) \to R$ such that $(g(x))^2 = x$ for all $x : D(f)$. By SQC, this corresponds to an element $\frac{p}{f^n} : R[X]_f$ with $(\frac{p}{f^n})^2 = X : R[X]_f$. We use lemma 1.3.6 together with the fact that $f(a_0)$ is invertible to get that $f : R[X]$ is regular, and therefore $p^2 = f^{2n}X : R[X]$. Considering this equation over $R^{\text{red}} = R/\sqrt{(0)}$ instead, we can show by induction that all coefficients of p and of f^n are nilpotent, which contradicts the invertibility of $f(a_0)$. □

Remark 3.4.5 Example 3.4.4 shows that the axioms we are using here are incompatible with a natural axiom that is true for the structure sheaf of the big étale topos, namely that R admits roots for unramifiable monic polynomials. The polynomial $X^2 - a$ is even separable for invertible a, assuming that 2 is invertible in R. To get rid of this last assumption, we can use the fact that either 2 or 3 is invertible in the local ring R and observe that the proof of example 3.4.4 works just the same for $X^3 - a$.

4 Topology of schemes

This section is not meant to be complete. Reduced schemes are defined in the work in progress draft [Che+23].

4.1 Closed subtypes

Definition 4.1.1 (a) A *closed proposition* is a proposition which is merely of the form $x_1 = 0 \wedge \dots \wedge x_n = 0$ for some elements $x_1, \dots, x_n \in R$.

(b) Let X be a type. A subtype $U : X \to \mathrm{Prop}$ is *closed* if for all $x : X$, the proposition $U(x)$ is closed.

Proposition 4.1.2 (using SQC) There is an order-reversing isomorphism of partial orders

$$\begin{aligned} \text{f.g.-ideals}(R) &\xrightarrow{\sim} \Omega_{cl} \\ I &\mapsto (I = (0)) \end{aligned}$$

between the partial order of finitely generated ideals of R and the partial order of closed propositions.

Proof For a finitely generated ideal $I = (x_1, \dots, x_n)$, the proposition $I = (0)$ is indeed a closed proposition, since it is equivalent to $x_1 = 0 \wedge \dots \wedge x_n = 0$. It is also evident that we get all closed propositions in this way. What remains to show is that

$$I = (0) \Rightarrow J = (0) \qquad \text{iff} \qquad J \subseteq I.$$

For this we use synthetic quasicoherence. Note that the set $\mathrm{Spec}\, R/I = \mathrm{Hom}_R(R/I, R)$ is a proposition (has at most one element), namely it is equivalent to the proposition $I = (0)$. Similarly, $\mathrm{Hom}_R(R/J, R/I)$ is a proposition and equivalent to $J \subseteq I$. But then our claim is just the equation

$$\mathrm{Hom}(\mathrm{Spec}\, R/I, \mathrm{Spec}\, R/J) = \mathrm{Hom}_R(R/J, R/I)$$

which holds by proposition 2.2.1, since R/I and R/J are finitely presented R-algebras if I and J are finitely generated ideals. □

Lemma 4.1.3 (using SQC, Loc, Z-choice) A closed subtype C of an affine scheme $X = \operatorname{Spec} A$ is an affine scheme with $C = \operatorname{Spec}(A/I)$ for a finitely generated ideal $I \subseteq A$.

Proof By Z-choice and boundedness, there is a cover $D(f_1), \dots, D(f_l)$, C such that on each $D(f_i)$, C is the vanishing set of functions

$$g_1, \dots, g_n : D(f_i) \to R.$$

So by lemma 1.3.8, there is a finitely generated ideal $I \subseteq A$, such that $A_{f_i} \cdot I$ is $(g_1, \dots, g_n)$ and $C = \operatorname{Spec} A/I$. □

There is an operation, which extends a closed subtype to some infinitesimal extend.

Definition 4.1.4 Let $X = \operatorname{Spec} A$ be affine and $C = \operatorname{Spec} A/I$ a closed subtype given by an finitely generated ideal $I \subseteq A$. Then the *n-th infinitesimal neighgorhood* of C in X is the closed subtype

$$C^n :\equiv \operatorname{Spec} A/(I^n) \subseteq \operatorname{Spec} A.$$

Explicitely, I^n is the ideal

$$I^n :\equiv \left\{ \sum_{i=1}^{m} \alpha_i x_{i,1} \cdot \dots \cdot x_{i,n} \mid x_{i,j} : I, \alpha_i : A \right\}.$$

So, by calculation, $(f_1, \dots, f_m)^n$ is generated by all n-fold products of the generators of the f_i.

There is an easy way describing the union of all the n-th neigbhorhoods of a closed subtype, using double negation:

Lemma 4.1.5 (using SQC, Loc, Z-choice) Let $X = \operatorname{Spec} A$ be affine, $C = \operatorname{Spec} A/I$ and $n : \mathbb{N}$, then $C^n \subseteq \neg\neg C$. The subtype $\neg\neg C$ is also called the *formal neighborhood* of C.

Proof Let C be given as $\operatorname{Spec} A/(f_1, \dots, f_l)$. Then, for any $x : X$

$$\begin{aligned} \neg\neg C(x) &= \neg\neg(x \in C) \\ &= \neg\neg(\forall i. f_i(x) = 0) \\ &= \forall_i \quad \neg\neg f_i(x) = 0 \\ &= \forall_i \quad f_i(x) \text{ is nilpotent} \end{aligned}$$

– and $C^n(x)$ means $\forall_i \quad f_i^n(x) = 0$. □

4.2 Open subtypes

While we usually drop the prefix "qc" in the definition below, one should keep in mind, that we only use a definition of quasi compact open subsets. The difference to general opens does not play a role so far, since we also only consider quasi compact schemes later.

Definition 4.2.1 (a) A proposition P is *(qc-)open*, if there merely are $f_1, \dots, f_n : R$, such that P is equivalent to one of the f_i being invertible.

(b) Let X be a type. A subtype $U : X \to \mathrm{Prop}$ is *(qc-)open*, if $U(x)$ is an open proposition for all $x : X$.

Proposition 4.2.2 (using SQC, Loc) A proposition P is open if and only if it is the negation of some closed proposition (definition 4.1.1).

Proof Indeed, by proposition 2.2.3, the proposition $\operatorname{inv}(f_1) \vee \dots \vee \operatorname{inv}(f_n)$ is the negation of $f_1 = 0 \wedge \dots \wedge f_n = 0$. □

Proposition 4.2.3 (using SQC, Loc) Let X be a type.

(a) The empty subtype is open in X.

(b) X is open in X.

(c) Finite intersections of open subtypes of X are open subtypes of X.

(d) Finite unions of open subtypes of X are open subtypes of X.

(e) Open subtypes are invariant under pointwise double-negation.
Axioms are only needed for the last statement.

In proposition 5.3.2 we will see that open subtypes of open subtypes of a scheme are open in that scheme. Which is equivalent to open propositions being closed under dependent sums.

Proof (of proposition 4.2.3) For unions, we can just append lists. For intersections, we note that invertibility of a product is equivalent to invertibility of both factors. Double-negation stability follows from proposition 4.2.2. ☐

Lemma 4.2.4 Let $f : X \to Y$ and $U : Y \to \mathrm{Prop}$ open, then the *preimage* $U \circ f : X \to \mathrm{Prop}$ is open.

Proof If $U(y)$ is an open proposition for all $y : Y$, then $U(f(x))$ is an open proposition for all $x : X$. ☐

Lemma 4.2.5 (using SQC, Loc) Let X be affine and $x : X$, then the proposition

$$x \neq y$$

is open for all $y : X$.

Proof We show a proposition, so we can assume $\iota : X \to \mathbb{A}^n$ is a subtype. Then for $x, y : X$, $x \neq y$ is equivalent to $\iota(x) \neq \iota(y)$. But for $x, y : \mathbb{A}^n$, $x \neq y$ is the open propostion that $x - y \neq 0$. ☐

The intersection of all open neighborhoods of a point in an affine scheme, is the formal neighborhood of the point. We will see in lemma 5.1.1, that this also holds for schemes.

Lemma 4.2.6 (using SQC, Loc) Let X be affine and $x : X$, then the proposition

$$\prod_{U : X \to \mathrm{Open}} U(x) \to U(y)$$

is equivalent to $\neg\neg(x = y)$.

Proof By proposition 4.2.3, $\neg\neg(x = y)$ implies $\prod_{U:X\to\mathrm{Open}} U(x) \to U(y)$. For the other implication, $\neg(x = y)$ is open lemma 4.2.5, so we get a contradiction. ☐

We now show that our two definitions (definition 3.1.4, definition 4.2.1) of open subtypes of an affine scheme are equivalent.

Theorem 4.2.7 (using Loc, SQC, Z-choice)
Let $X = \operatorname{Spec} A$ and $U : X \to \mathrm{Prop}$ be an open subtype, then U is affine open, i.e. there merely are $f_1, \dots, f_n : X \to R$ such that $U = D(f_1, \dots, f_n)$.

Proof Let $L(x)$ be the type of finite lists of elements of R, such that one of them being invertible is equivalent to $U(x)$. By assumption, we know

$$\prod_{x:X} \|L(x)\|.$$

So by Z-choice, we have $s_i : \prod_{x:D(f_i)} L(x)$. We compose with the length function for lists to get functions $l_i : D(f_i) \to \mathbb{N}$. By theorem 3.3.3, the l_i are bounded. Since we are proving a proposition, we can assume we have actual bounds $b_i : \mathbb{N}$. So we have functions $\tilde{s_i} : D(f_i) \to R^{b_i}$. ☐

This allows us to transfer one important lemma from affine-opens to qc-opens. The subtlety of the following is that while it is clear that the intersection of two qc-opens on a type, which are *globally* defined is open again, it is not clear, that the same holds, if one qc-open is only defined on the other.

Lemma 4.2.8 (using Loc, SQC, Z-choice) Let X be a scheme, $U \subseteq X$ qc-open in X and $V \subseteq U$ qc-open in U, then V is qc-open in X.

Proof Let $X_i = \operatorname{Spec} A_i$ be a finite affine cover of X. It is enough to show, that the restriction V_i of V to X_i is qc-open. $U_i :\equiv X_i \cap U$ is qc-open in X_i, since X_i is qc-open. By theorem 4.2.7, U_i is affine-open in X_i, so $U_i = D(f_1, \dots, f_n)$. $V_i \cap D(f_j)$ is affine-open in $D(f_j)$, so by lemma 3.1.6, $V_i \cap D(f_j)$ is affine-open in X_i. This implies $V_i \cap D(f_j)$ is qc-open in X_i and so is $V_i = \bigcup_j V_i \cap D(f_j)$. ☐

Lemma 4.2.9 (using Loc, SQC, Z-choice) (a) qc-open propositions are closed under dependent sums: if P : Open and $U : P \to \text{Open}$, then the proposition $\sum_{x:P} U(x)$ is also open.

(b) Let X be a type. Any open subtype of an open subtype of X is an open subtype of X.

Proof (a) Apply lemma 4.2.8 to the point $\operatorname{Spec} R$.

(b) Apply the above pointwise. □

Remark 4.2.10 Lemma 4.2.9 means that the (qc-) open propositions constitute a *dominance* in the sense of [Ros86].

The following fact about the interaction of closed and open propositions is due to David Wärn.

Lemma 4.2.11 Let P and Q be propositions with P closed and Q open. Then $P \to Q$ is equivalent to $\neg P \vee Q$.

Proof We can assume $P = (f_1 = \cdots = f_n = 0)$ and $Q = (\text{inv}(g_1) \vee \cdots \vee \text{inv}(g_m))$. Then we have:

$$\begin{aligned}
(P \to Q) &= && \text{proposition 2.2.3 for } g_1, \ldots, g_m \\
(P \to \neg(g_1 = \cdots = g_m = 0)) &= && \\
\neg(f_1 = \cdots = f_n = g_1 = \cdots = g_m = 0) &= && \text{proposition 2.2.3 for } f_1, \ldots, f_n, g_1, \ldots, g_m \\
(\text{inv}(f_1) \vee \cdots \vee \text{inv}(f_n) \vee \text{inv}(g_1) \vee \cdots \vee \text{inv}(g_m)) &= && \text{proposition 2.2.3 for } f_1, \ldots, f_n \\
\neg P \vee Q &&&
\end{aligned}$$

4.3 Definition of schemes

The following definition *does not* define schemes in general, but something which is expected to correspond to quasi-compact, quasi separated schemes, locally of finite type externally.

Definition 4.3.1 A type X is a *(qc-)scheme* if there merely is a cover by finitely many open subtypes $U_i : X \to \text{Prop}$, such that each of the U_i is affine.

Definition 4.3.2 We denote the *type of schemes* with Sch_{qc}.

Zariski-choice Z-choice extends to schemes:

Proposition 4.3.3 (using Z-choice) Let X be a scheme and $P : X \to \text{Type}$ with $\prod_{x:X} \|P(x)\|$, then there merely is a cover U_i by standard opens of the affine parts of X, such that there are $s_i : \prod_{x:U_i} P(x)$ for all i.

4.4 Connectedness

The following is in conflict with the usual use of the word "connected" in homotopy type theory.

Definition 4.4.1 A pointed type X is called *connected*, if the following equivalent statements hold:
(i) Any function $X \to \text{Bool}$ is constant.

(ii) Any detachable subset is X or $\emptyset$.

Proposition 4.4.2 (using SQC, Loc) The set $\mathbb{A}^1$ is connected, that is, every function $f : \mathbb{A}^1 \to \text{Bool}$ is constant.

Proof We embed Bool into R as the subset $\{0, 1\} \subseteq R$. (We have $0 \neq 1$ in R by (Loc).) Then we have a function $\widetilde{f} : \mathbb{A}^1 \to R$ and we can assume $\widetilde{f}(0) = 0$. Note that $\widetilde{f}$ is an idempotent element of the algebra $R^{\mathbb{A}^1}$, since all its values are idempotent elements of R. By (SQC), $\widetilde{f}$ is given by an idempotent polynomial $p \in R[X]$ with $p(0) = 0$. But from this follows $p = 0$: we can factorize $p = Xq$ and then calculate $p = p^n = X^n q^n$ to see that all coefficients of p are zero. □

A connected scheme, that is covered by its point and everything except the point, is already trivial.

Corollary 4.4.3 Let X be a connected scheme and

$$\prod_{x:X} x = * \vee x \neq *.$$

Then X is contractible.

Proof Assume $\prod_{x:X} x = * \vee x \neq *$. Since for any proposition P, $P + \neg P$ is a proposition, we have $\prod_{x:X} x = * + x \neq *$ and there is a map to Bool from any binary copdoduct. So we have a map $X \to \mathrm{Bool}$ which decides if a general $x : X$ is the point $*$ or not. By connectedness of X, this map is constant, but we know $* = *$, so $x = *$ for all x. □

Corollary 4.4.4 (using SQC, Loc) $\neg(\prod_{x:\mathbb{A}^1} x = 0 \vee x \neq 0)$

Proof By corollary 4.4.3 and by the connectedness of $\mathbb{A}^1$ (proposition 4.4.2), we can show from $\prod_{x:\mathbb{A}^1} x = 0 \vee x \neq 0$ that $\mathbb{A}^1$ is contractible. This contradicts $1 \neq 0$. □

Example 4.4.5 The ring R is a local ring, so we have $\Pi_{x\in R} \|\mathrm{inv}(x) \vee \mathrm{inv}(1-x)\|$, but we can prove that the statement without the propositional truncation is false:

$$\neg\Pi_{x\in R}(\mathrm{inv}(x) \amalg \mathrm{inv}(1-x)).$$

Namely, a witness of $\Pi_{x\in R}(\mathrm{inv}(x) \amalg \mathrm{inv}(1-x))$ is equivalently a function $f : R \to \mathrm{Bool}$ with the property that

$$\text{if } f(x) \text{ then } \mathrm{inv}(x) \text{ else } \mathrm{inv}(1-x).$$

But by *proposition* 4.4.2, the function f must be constant, contradicting the fact that $\neg\mathrm{inv}(x)$ for $x = 0$ and $\neg\mathrm{inv}(1-x)$ for $x = 1$.

In particular, not every type family $B : \mathbb{A}^1 \to \mathcal{U}$ with $\Pi_{x:\mathbb{A}^1} \|B(x)\|$ merely admits a choice function $\Pi_{x:\mathbb{A}^1} B(x)$.

4.5 Compactness properties

Theorem 3.3.3 can be read as a compactness property for countable disjoint open coverings of affine schemes, since functions $f : \mathrm{Spec}\, A \to \mathbb{N}$ correspond to decompositions $\mathrm{Spec}\, A = \sum_{n:\mathbb{N}} U_n$, and the subsets $U_n \subseteq X$ are automatically open because they are detachable.

The following example shows that we can not expect all affine schemes to be compact with respect to arbitrary set-indexed open coverings.

Example 4.5.1 For A a finitely presented R-algebra, consider the open cover $(U_i)_{i\in I}$, where the index set is $I = \mathrm{Spec}\, A$ and for each i we set $U_i = \mathrm{Spec}\, A = D(1)$. This indeed covers all points of $\mathrm{Spec}\, A$, since for every $x \in \mathrm{Spec}\, A$ we clearly have $x \in U_x$. To give a finite subcover of this cover, however, means to give a natural number n and a function $\mathrm{Fin}\, n \to \mathrm{Spec}\, A$ with the property that $\mathrm{Spec}\, A$ is empty if $n = 0$. In essence, it means to decide whether $\mathrm{Spec}\, A$ is inhabited or not. We claim that this is not possible for all finitely presented R-algebras:

$$\neg(\Pi_{A:f.p.R\text{-Alg}} \|\mathrm{Spec}\, A \amalg \neg\, \mathrm{Spec}\, A\|).$$

Indeed, for $A = R/(x)$, the proposition $\|\mathrm{Spec}\, A \amalg \neg\, \mathrm{Spec}\, A\|$ means $x = 0 \vee x \neq 0$, and we saw in **??** that this is not true for all $x \in R$.

There is, however, a notion of compactness, which seems to correspond to completeness and therefore also leads to a notion of properness, which is treated in [23].

4.6 Dense subtypes

Algebraic preparation:

Lemma 4.6.1 If $P : R[X]$ and we have $P \neq 0$, or equivalently, that merely some coefficient of P is non-zero, then P is nilregular.

Proof If P is non-zero, its content $c(P)$ is top. For any $Q : R[X]$ with $P \cdot Q$ nilpotent, by [LQ15][Theorem III.2.1] we also have $c(P \cdot Q) = c(P) \wedge c(Q) = c(Q)$ is bottom. So Q is nilpotent. □

In a lattice, an element a can be called dense if $a \wedge b = \bot$ implies $b = \bot$. We apply this definition to the lattice of open subtypes of a type X, but generalize it to allow for non-open dense subtypes too.

Definition 4.6.2 A subtype $A \subseteq X$ is called *dense*, if for all open subtypes $V \subseteq X$ such that $V \cap A = \emptyset$, we have $V = \emptyset$.

Lemma 4.6.3 Let X be a type.

(a) If $D \subseteq X$ is dense, then $X \neq \emptyset$ implies $D \neq \emptyset$.

(b) Let $D \subseteq X$ be dense and open and let $E \subseteq X$ be dense, then $D \cap E$ is dense.

(c) Let $D \subseteq X$ be dense and $E \subseteq X$ any subtype, then $D \cup E$ is dense.

Proof (a) Assume $D \subseteq X$ is dense and empty. Then $X \cap D = \emptyset$ and by denseness of D, the open subtype $X \subseteq X$ is empty, which contradicts $X \neq \emptyset$.

(b) Let $V \subseteq X$ be open with $V \cap D \cap E = \emptyset$. Since $V \cap D$ is open and E is dense, we get $V \cap D = \emptyset$, and by denseness of D, we get $V = \emptyset$.

(c) Let $V \subseteq X$ be open with

$$\emptyset = V \cap (D \cup E) = (V \cap D) \cup (V \cap E).$$

This implies $V \cap D = \emptyset$, so $V = \emptyset$. □

Being dense is double negation stable — which has the practical implication, that we can "open" double-negated statements when showing denseness.

Proposition 4.6.4 Being dense is $\neg\neg$-stable: if a subtype $D \subseteq X$ is not not dense, then it is dense.

Proof We use general facts about modalities. $V = \emptyset$ is a pointwise negated statement and therefore $\neg\neg$-stable. Since the proposition that D is dense is a $\prod$-type with values in $V = \emptyset$, it is also $\neg\neg$-stable. □

Lemma 4.6.5 (using SQC, Loc, Z-choice) Let X be a type, let $U \subseteq X$ be an open subtype and let $D \subseteq X$ be dense. Then $U \cap D$ is a dense subtype of U.

Proof Let $V \subseteq U$ be open with $V \cap (U \cap D) = \emptyset$. Using lemma 4.2.9, $V \subseteq X$ is open and $V \cap D = V \cap (U \cap D) = \emptyset$. So $V = \emptyset$ since D is dense in X. □

Being dense is a local property in the following sense:

Lemma 4.6.6 (using SQC, Loc, Z-choice) Let X be a type and $U_i \subseteq X$ be open subtypes for $i : I$ such that $\bigcup_{i:I} U_i = X$. Then $A \subseteq X$ is dense, if and only if, $A \cap U_i$ is dense in U_i for every i.

Proof Let all $A \cap U_i$ be dense. To show that A is dense, let $V \subseteq X$ be open and $V \cap A = \emptyset$. Then $\emptyset = V \cap A = \bigcup_{i:I} (V \cap U_i) \cap (A \cap U_i)$, so $(V \cap U_i) \cap (A \cap U_i) = \emptyset$ for all $i : I$. But $V \cap U_i$ is open in U_i, so by assumption, $V \cap U_i = \emptyset$ for all $i : I$. So $V = \bigcup_{i:I} V \cap U_i = \emptyset$ and A is dense.

The other direction follows from lemma 4.6.5. □

We will now characterize dense open subsets of affine schemes.

Definition 4.6.7 Let A be a commutative ring.

(a) An element $r : A$ is *nilregular*, if for all $x : A$, such that rx is nilpotent, x is nilpotent.

(b) A list of elements $r_1, \ldots, r_n : A$ is *jointly nilregular*, if for all $x : A$, such that all $r_i x$ are nilpotent, x is nilpotent.

Proposition 4.6.8 Any regular (definition 1.3.3) element $r : A$ is nilregular.

Lemma 4.6.9 (using SQC, Loc, Z-choice) Let $X = \operatorname{Spec} A$ be affine and $U \subseteq X$ open. Then U is dense, if and only if, $U = D(r_1, \ldots, r_n)$, with jointly nilregular $r_1, \ldots, r_n : A$.

Proof Let $U \subseteq X$ be dense and open. By theorem 4.2.7, there are $r_1, \ldots, r_n : A$ such that $U = D(r_1, \ldots, r_n)$. Let $x : A$ such that all $r_i x$ are nilpotent. By lemma 3.1.7, this implies $D(r_i x) = \emptyset$ for all i. Since $D(x) \cap D(r_1, \ldots, r_n) = D(r_1 x) \cup \cdots \cup D(r_n x) = \emptyset$, this implies $D(x) = \emptyset$. Therefore x is nilpotent and the r_i are jointly nilregular.

Now let $U = D(r_1, \ldots, r_n)$ with jointly nilregular $r_1, \ldots, r_n : A$. Without loss of generality, let $V = D(f)$ and $D(f) \cap U = \emptyset$. Then $D(r_1 f) \cup \cdots \cup D(r_n f) = \emptyset$, so $D(r_i f) = \emptyset$ for all i. This means $r_i f$ is nilpotent and therefore, f is nilpotent and $D(f) = \emptyset$. □

Corollary 4.6.10 (using SQC, Loc, Z-choice) The only dense open subset of $1 = \operatorname{Spec} R$ is 1.

Proof Let $U \subseteq 1$ be dense and open. By lemma 4.6.9, there are jointly nilregular $r_1, \ldots, r_n : R$, such that $U = D(r_1, \ldots, r_n)$. But jointly nilregular entails, that one of the r_i is invertible, so $U = 1$. □

Theorem 4.6.11 (using SQC, Loc, Z-choice)
Let X be a scheme. An open subtype $U \subseteq X$ is dense, if and only if, there is an open affine cover $U_i = \operatorname{Spec} A_i$ and $U \cap U_i = D(r_{i1}, \ldots, r_{in_i})$ with jointly nilregular $r_{i1}, \ldots, r_{in_i} : A_i$ for all i.

Proof By lemma 4.6.9 and lemma 4.6.6. □

Classicly, one possible definition of a dense subset is that the closure is the whole space. We will see an approximation to that in lemma 5.3.6. There are lots of examples of non-trivial dense subsets. For example, the next section will contain a proof, that any non-empty open subset of $\mathbb{A}^1$ is dense.

4.7 Closed dense subtypes

This section is due to Hugo Moeneclaey.

Lemma 4.7.1 For any type X, a closed subtype $C : X \to \text{Prop}$ is dense if and only if:

$$\prod_{x:X} \neg\neg C(x)$$

Proof Assume C a closed subtype of X. If C is dense, as $\neg C$ is open and $\neg C \cap C = \emptyset$, we have that $\neg C = \emptyset$ which is precisely what we want.

Conversely assume that for all $x : X$ we have $\neg\neg C(x)$. Let U be an open subtype of X such that $U \cap C = \emptyset$. Then for any $x : X$ we have $\neg(C(x) \wedge U(x))$ as well as $\neg\neg C(x)$, so that we have $\neg U(x)$. So we have $U = \emptyset$ and C is indeed dense. □

Corollary 4.7.2 The type of closed propositions C such that $\neg\neg C$ classifies closed dense subtypes.

Proposition 4.7.3 (using SQC, Loc) A closed subscheme $\operatorname{Spec}(A/I)$ of an affine scheme $\operatorname{Spec}(A)$ is dense if and only if I is nilpotent.

Proof Assume $\operatorname{Spec}(A/I) \subset \operatorname{Spec}(A)$ dense. For any $f : I$, we have $\operatorname{Spec}(A/I) \cap D(f) = \emptyset$ and $D(f)$ open so that $D(f) = \emptyset$ and f is nilpotent.

Conversely, let I be a finitely generated nilpotent ideal in A generated by $f_1, \cdots, f_n$. Then for all $x : \operatorname{Spec}(A)$, we have $x \in \operatorname{Spec}(A/I)$ if and only if:

$$f_1(x) = 0 \wedge \cdots \wedge f_n(x) = 0$$

But as $f_1, \cdots, f_n$ are nilpotent we have:

$$\neg\neg(f_1(x) = 0) \wedge \cdots \wedge \neg\neg(f_n(x) = 0)$$

so that:

$$\neg\neg(x \in \operatorname{Spec}(A/I))$$

and $\operatorname{Spec}(A/I)$ is dense by lemma 4.7.1. □

4.8 Irreducible and reducible types

We start with the notion of reducible types and will then pass to the negation of this concept, to irreducible types.

Definition 4.8.1 A type is called *reducible*, if there are two disjoint, inhabited open subtypes.

Proposition 4.8.2 The scheme $\operatorname{Spec} R[X, Y]/(XY)$ is reducible.

Proof We take the subsets $D(X), D(Y) \subseteq \operatorname{Spec} R[X, Y]/(XY)$. Then

$$D(X) \cap D(Y) = \{\, (x, y) \mid xy = 0 \wedge x \neq 0 \wedge y \neq 0 \,\} = \emptyset.$$

And $(1, 0) \in D(X)$, $(0, 1) \in D(Y)$. □

Definition 4.8.3 A type X is called *irreducible*, if the following equivalent propositions hold:

(i) X is not reducible.

(ii) Any non-empty open $U \subseteq X$ is dense.

(iii) For any open disjoint $U, V \subseteq X$ such that $U \neq \emptyset$, we have $V = \emptyset$.

Proposition 4.8.4 Being irreducible is $\neg\neg$-stable.

Proof By the definition as not reducible, or by proposition 4.6.4. ☐

Example 4.8.5 Every proposition is an irreducible type, since any two inhabited subtypes intersect.

Proposition 4.8.6 (using SQC, Loc, Z-choice) $\mathbb{A}^1$ is irreducible.

Proof Let $U \subseteq \mathbb{A}^1$ with $U \neq \emptyset$. We have to show that U is dense. Let $U = D(f_1, \dots, f_n)$. We merely have a bound for the degree of each of the $f_i : R[X]$, so we can concatenate all coefficients of the f_i and, since $U \neq \emptyset$, we know that vector is not the zero-vector. So one of the f_i is nilregular by lemma 4.6.1. In particular, the elements $f_1, \dots, f_n$ are jointly nilregular, so U is dense by lemma 4.6.9. ☐

Example 4.8.7 The scheme $\operatorname{Spec} R[X,Y]/(XY)$ is not irreducible.

Remark 4.8.8 In a classical setting, reducibility and irreducibility are usually defined in terms of closed subsets instead of open subsets. However, this does not give the correct notion in our setting, as the example $\operatorname{Spec} R[X,Y]/(XY)$ shows: this scheme is not the union of the closed subsets $V(X)$ and $V(Y)$.

We will now explore the relation of connectedness and irreducibility. It is not the case, that any open dense subtype of a connected scheme is connected:

Example 4.8.9 Let us first show, that $V(XY) \subseteq \mathbb{A}^2$ is connected. Let $f : V(XY) \to \mathrm{Bool}$ be a function and assume without loss of generality, that $f(0,0) = 1$. Then the restriction of $D(f)$ to both, $V(X)$ or $V(Y)$ is dense. Since $f(x) = 1$ is closed and holds for $x : D(f)$, $f(x) = 1$ holds not not for all $x : V(XY)$, which is enough.

The open subtype $D(X,Y) \subseteq V(XY) \subseteq \mathbb{A}^2$ is not connected. This is witnessed by the function

$$\frac{X}{X+Y}.$$

Proposition 4.8.10 (using no axioms) Any irreducible pointed type is connected.

Proof Let X be an irreducible pointed type and let a decomposition into detachable subsets $X = U \sqcup V$ be given. In particular, U and V are open subsets, and we can assume that the base point of X lies in U. But then U is dense since X is irreducible, so we have $V = \emptyset$ and $U = X$. ☐

Proposition 4.8.11 (using Loc, SQC, Z-choice) Let X be an irreducible type and $U \subseteq X$ an open subtype. Then U is also irreducible.

Proof Let $V, W \subseteq U$ be open subtypes with $V \cap W = \emptyset$. Assume that both V and W are nonempty, now we have to show a contradiction. By lemma 4.2.9, V and W are also open subsets of X, so we indeed get a contradiction from the fact that X is irreducible. ☐

Lemma 4.8.12 Let X and Y be irreducible types. Then $X \times Y$ is irreducible.

The following proof is due to David Wärn.

Proof Let $U, V \subseteq X \times Y$ be disjoint open subsets with $(a,b) \in U$, $(c,d) \in V$. Consider the subtypes

$$U_a :\equiv \{\, y : Y \mid (a,y) \in U \,\}$$

and

$$V_c :\equiv \{\, y : Y \mid (c,y) \in V \,\}.$$

These are two inhabited open subtypes of Y, so they can not be disjoint. Since we want to show a contradiction, we can assume we have $e \in U_a \cap V_c$. But then

$$U_e :\equiv \{\, x : X \mid (x,e) \in U \,\}$$

and

$$V_e :\equiv \{\, x : X \mid (x,e) \in V \,\}$$

are open subtypes of X which are disjoint since U and V are disjoint, and inhabited by a respectively c. This contradicts the irreducibility of X. □

Lemma 4.8.13 Let X be irreducible and $f : X \to Y$ be surjective, then Y is irreducible.

Proof Using the definition with disjoint opens. □

We will see in proposition 6.1.8, that the projective n-space $\mathbb{P}^n$ is irreducible.

4.9 Separated types and apartness

Proposition 4.9.3 was found and proven together with Marc Nieper-Wißkirchen and Ingo Blechschmidt.

Definition 4.9.1 A type X is *separated*, if for all $x, y : X$ the type $x = y$ is a closed proposition, that is, the diagonal $X \to X \times X$ is the embedding of a closed subtype.

Definition 4.9.2 An *apartness relation* on X is a relation $\# : X \to X \to \mathrm{Prop}$, such that it is
(i) irreflexive: $\prod_{x:X} \neg(x \# x)$
(ii) symmetric: $\prod_{x,y:X} x \# y \to y \# x$
(iii) and cotransitive: $\prod_{x,y,z:X} x \# z \to x \# y \vee y \# z$.

Proposition 4.9.3 If X is a separated scheme, then inequality is an apartness relation.

Proof MISSING □

Proposition 4.9.4 (using SQC, Loc, Z-choice) Let X be a separated scheme and U, V be open affine in X. Then $U \cap V$ is affine.

Proof $U \cap V$ is equivalently the closed subtype $\{(x,y) : U \times V \mid x = y\}$. $U \times V$ is affine by lemma 3.2.2 and a closed subtype of an affine scheme is affine by lemma 4.1.3. □

5 Schemes

In definition 4.3.1 we defined a scheme to be a type X such that X may be covered by finitely many open affine subtypes. In this section, we will present general properies of schemes and a couple of common constructions for schemes.

5.1 General Properties

Lemma 5.1.1 (using SQC, Loc) Let X be a scheme and $x : X$, then for all $y : X$ the proposition

$$\prod_{U:X\to\mathrm{Open}} U(x) \to U(y)$$

is equivalent to $\neg\neg(x = y)$.

Proof By proposition 4.2.3, open proposition are always double-negation stable, which settles one implication. For the implication

$$\left(\prod_{U:X\to\mathrm{Open}} U(x) \to U(y)\right) \Rightarrow \neg\neg(x = y)$$

we can assume that x and y are both inside an open affine U and use that the statement holds for affine schemes by lemma 4.2.6. □

5.2 Glueing

Proposition 5.2.1 (using Loc, SQC, Z-choice) Let X, Y be schemes and $f : U \to X$, $g : U \to Y$ be embeddings with open images in X and Y, then the pushout of f and g is a scheme.

Proof As we noted in **??**, such a pushout is always 0-truncated. Let $U_1, \dots, U_n$ be a cover of X and $V_1, \dots, V_m$ be a cover of Y. By lemma 4.2.8, $U_i \cap U$ is open in Y, so we can use (large) pushout-recursion to construct a subtype $\tilde{U_i}$, which is open in the pushout and restricts to U_i on X and $U_i \cap U$ on Y. Symetrically we define $\tilde{V_i}$ and in total get an open finite cover of the pushout. The pieces of this new cover are equivalent to their counterparts in the covers of X and Y, so they are affine as well. □

5.3 Subschemes

Definition 5.3.1 Let X be a scheme. A *subscheme* of X is a subtype $Y : X \to \text{Prop}$, such that $\sum Y$ is a scheme.

Proposition 5.3.2 (using Loc, SQC, Z-choice) Any open subtype of a scheme is a scheme.

Proof Using theorem 4.2.7. □

Proposition 5.3.3 (using SQC, Loc, Z-choice) Any closed subtype $A : X \to \text{Prop}$ of a scheme X is a scheme.

Proof Any open subtype of X is also open in A. So it is enough to show, that any affine open U_i of X, has affine intersection with A. But $U_i \cap A$ is closed in U_i and therefore affine by lemma 4.1.3. □

We can extend the operation from definition 4.1.4 to schemes:

Definition 5.3.4 (using SQC, Loc, Z-choice) Let X be a scheme and $C \subseteq X$ a closed subscheme. Then C^n is the closed subscheme of X, defined locally as in definition 4.1.4.

Proof We need the axioms to locally get ideals that generate the closed subscheme. We need to show that the construction can be done locally, but this is the case, since for any open affine U, $(C \cap U)^n \subseteq \neg\neg(C \cap U) \subseteq U$ by **??**. □

Lemma 5.3.5 (using SQC, Loc, Z-choice) Let $U \subseteq X$ be a dense open subtype of a scheme. For any closed subtype V containing U, there merely is an $n : \mathbb{N}$, such that $V^n = X$.

Proof It is enough to do the construction for an open affine $W = \text{Spec}\, A$, where $V \cap U = \text{Spec}\, A/(f_1, \dots, f_n)$ and $U = D(g_1, \dots, g_l)$. By theorem 4.6.11 we can assume the $g_1, \dots, g_l$ are jointly nilregular. For any f_i we know $f_i \cdot g_j$ is nilpotent, since

$$\neg D(f_i g_j) = \neg\{x : W \mid f_i g_i(x) \text{ invertible}\} = \emptyset,$$

since if $f_i g_j(x)$ is invertible, then $g_j(x)$ is invertible, but then, we are in U and $f_i(x)$ has to be zero, which contradicts its invertibility.

By the joint nilregularity of the g_j, f_i is nilpotent, so $f_i^n = 0$ and $V^n = W$. □

In the situation of a clopen subset, we get the classical equality:

Lemma 5.3.6 (using SQC, Loc, Z-choice) Let $U \subseteq X$ be a dense open and closed subtype of a scheme, then $U = X$

Proof By lemma 5.3.5, $U^n = X$. By lemma 4.1.5, we have

$$X \subseteq U^n \subseteq \neg\neg U = U.$$ □

Theorem 5.3.7 (using SQC, Loc, Z-choice)
The ring R is not coherent, i.e. it is not the case, that all finitely generated ideals in R are finitely presented.

Proof We will show, that it is not the case, that any R-module map $R \to R$ has a finitely generated kernel. Every R-linear map $\varphi : R \to R$ is of the form $\varphi_x(z) = xz$ for some $x : R$, namely $x :\equiv \varphi(1)$. Assume it is always possible to find generators $y_1, \ldots, y_n : R$ of the kernel of φ_x. That means there is a map

$$c : \prod_{x:R} \left(\exists_{y_1,\ldots,y_n:R} \prod_{z:R} \left(\varphi_x(z) = 0 \text{ iff } \exists_{\lambda_1,\ldots,\lambda_n:R} z = \sum_{i=1}^{n} \lambda_i y_i \right)\right).$$

By Z-choice and boundedness (theorem 3.3.3), we translate the first "$\exists$" to a function $g : D(f) \to R^n$ on a neighborhood $D(f) \subseteq R$ of $0 : R$. We know that if x is invertible, then $\ker(\varphi_x) = (0)$, which means $y_i = 0$ for all i. So $g(x)$ must be the 0-vector for all $x : D(f) \cap D(X)$. Since $\mathbb{A}^1$ is irreducible by proposition 4.8.6, $D(X)$ and $D(f)$ are both already dense by being non-empty. By lemma 4.6.5 $D(f) \cap D(X)$ is dense in $D(f)$, so by lemma 5.3.5 applied to $V(g_i) \subseteq D(f)$, the entries $g_i(x)$ of $g(x)$ must be nilpotent for all $x : D(f)$. But this is a contradiction, since for $x = 0$, the kernel of φ_x is R and there must be an invertible entry in $g(x)$. □

5.4 Equality types

Lemma 5.4.1 Let X be an affine scheme and $x, y : X$, then $x =_X y$ is an affine scheme and $((x, y) : X \times X) \mapsto x =_X y$ is a closed subtype of $X \times X$.

Proof Any affine scheme is merely embedded into $\mathbb{A}^n$ for some $n : \mathbb{N}$. The proposition $x = y$ for elements $x, y : \mathbb{A}^n$ is equivalent to $x - y = 0$, which is equivalent to all entries of this vector being zero. The latter is a closed proposition. □

Proposition 5.4.2 (using SQC, Loc, Z-choice) Let X be a scheme. The equality type $x =_X y$ is a scheme for all $x, y : X$.

Proof Let $x, y : X$ and $U \subseteq X$ be an affine open containing x. Then $U(y) \wedge x = y$ is equivalent to $x = y$, so it is enough to show that $U(y) \wedge x = y$ is a scheme. As a open subscheme of the point, $U(y)$ is a scheme and $(x : U(y)) \mapsto x = y$ defines a closed subtype by lemma 5.4.1. But this closed subtype is a scheme by proposition 5.3.3. □

5.5 Dependent sums

Theorem 5.5.1 (using Loc, SQC, Z-choice)
Let X be a scheme and for any $x : X$, let Y_x be a scheme. Then the dependent sum

$$((x : X) \times Y_x) \equiv \sum_{x:X} Y_x$$

is a scheme.

Proof We start with an affine $X = \operatorname{Spec} A$ and $Y_x = \operatorname{Spec} B_x$. Locally on $U_i = D(f_i)$, for a Zariski-cover $f_1, \ldots, f_l$ of X, we have $B_x = \operatorname{Spec} R[X_1, \ldots, X_{n_i}]/(g_{i,x,1}, \ldots, g_{i,x,m_i})$ with polynomials $g_{i,x,j}$. In other words, B_x is the closed subtype of $\mathbb{A}^{n_i}$ where the functions $g_{i,x,1}, \ldots, g_{i,x,m_i}$ vanish. By lemma 3.2.2, the product

$$V_i :\equiv U_i \times \mathbb{A}^{n_i}$$

is affine. The type $(x : U_i) \times \operatorname{Spec} B_x \subseteq V_i$ is affine, since it is the zero set of the functions

$$((x, y) : V_i) \mapsto g_{i,x,j}(y)$$

Furthermore, $W_i :\equiv (x : U_i) \times \operatorname{Spec} B_x$ is open in $(x : X) \times Y_x$, since $W_i(x)$ is equivalent to $U_i(\pi_1(x))$, which is an open proposition.

This settles the affine case. We will now assume, that X and all Y_x are general schemes. We pass again to a cover of X by affine open $U_1, \ldots, U_n$. We can choose the latter cover, such that for each i and $x : U_i$, the $Y_{\pi_1(x)}$ are covered by l_i many open affine pieces $V_{i,x,1}, \ldots, V_{i,x,l_i}$ (by theorem 3.3.3). Then $W_{i,j} :\equiv (x : U_i) \times V_{i,x,j}$ is affine by what we established above. It is also open. To see this, let $(x, y) : ((x : X) \times Y_x)$. We want to show, that (x, y) being in $W_{i,j}$ is an open proposition. We have to be a bit careful, since the open proposition $V_{i,x,j}$ is only defined, for $x : U_i$. So the proposition we are after is $(z : U_i(x, y)) \times V_{i,z,j}(y)$. But this proposition is open by lemma 4.2.9. □

Corollary 5.5.2 Let X be a scheme. For any other scheme Y and any map $f : Y \to X$, the fiber map $(x : X) \mapsto \mathrm{fib}_f(x)$ has values in the type of schemes Sch_{qc}. Mapping maps of schemes to their fiber maps, is an equivalence of types

$$\left(\sum_{Y:\mathrm{Sch}_{qc}} (Y \to X)\right) \simeq (X \to \mathrm{Sch}_{qc}).$$

Proof By univalence, there is an equivalence

$$\left(\sum_{Y:\mathrm{Type}} (Y \to X)\right) \simeq (X \to \mathrm{Type}).$$

From left to right, the equivalence is given by turning a $f : Y \to X$ into $x \mapsto \mathrm{fib}_f(x)$, from right to left is given by taking the depedent sum. So we just have to note, that both constructions preserve schemes. From left to right, this is theorem 5.6.1, from right to left, this is theorem 5.5.1. □

Subschemes are classified by propositional schemes:

Corollary 5.5.3 Let X be a scheme. $Y : X \to \mathrm{Prop}$ is a subscheme, if and only if Y_x is a scheme for all $x : X$.

Proof Restriction of corollary 5.5.2. □

5.6 Pullbacks of Schemes

In this section, we will show in two different ways, that the pullback of a cospan of schemes is a scheme. The first poof is very short and reuses what we proved about equality and sigma-types, the second proof is more direct, uses the proof of the affine case lemma 3.2.2 and is along the lines of what one might find in an algebraic geometry textbook.

Theorem 5.6.1 (using Loc, SQC, Z-choice)
Let

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

be schemes, then the *pullback* $X \times_Z Y$ is also a scheme.

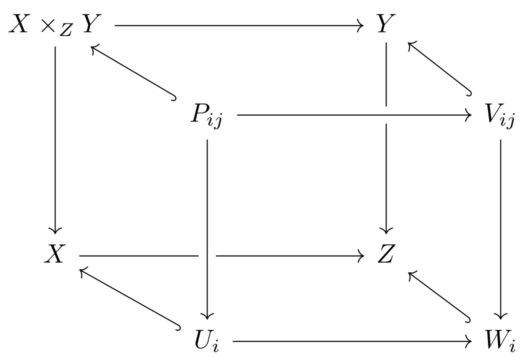
Proof The type $X \times_Z Y$ is given as the following, interated dependent sum:

$$\sum_{x:X} \sum_{y:Y} f(x) = g(y).$$

The innermost type, $f(x) = g(y)$ is the equality type in the scheme Z and by proposition 5.4.2 a scheme. By applying theorem 5.5.1 twice, we prove that the itereated dependent sum is a scheme. □

We conclude with a construction, analogous to the classical treatment:

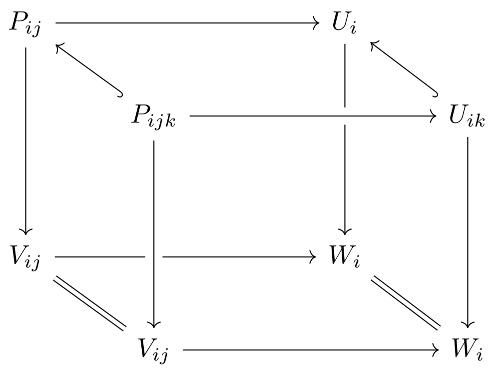
Proof (alternative proof of theorem 5.6.1) Let $W_1, \dots, W_n$ be a finite affine cover of Z. The preimages of W_i under f and g are open and covered by fintely many affine open U_{ik} and V_{ij} by proposition 5.3.2. This leads to the following diagram:



where the front and bottom square are pullbacks by definition. By pullback-pasting, the top is also a pullback, so all diagonal maps are embeddings.

P_{ij} is open, since it is a preimage of V_{ij} (lemma 4.2.4), which is open in Y by lemma 4.2.8. It remains to show, that the P_{ij} cover $X \times_Z Y$ and that P_{ij} is a scheme. Let $x : X \times_Z Y$. For the image w of x in W, there merely is an i such that w is in W_i. The image of x in V_i merely lies in some V_{ij}, so x is in P_{ij}.

We proceed by showing that P_{ij} is a scheme. Let U_{ik} be a part of the finite affine cover of U_i. We repeat part of what we just did:



So by lemma 3.2.2, P_{ijk} is affine. Repetition of the above shows, that the P_{ijk} are open and cover P_{ij}. □

6 Projective space

6.1 Construction of projective spaces

We give two definitions of projective space, which differ only in size.

Definition 6.1.1 (a) An n-dimensional *R-vector space* is an R-module V, such that $\|V = R^n\|$.

(b) We write $R\text{-Vect}_n$ for the type of these vector spaces and $V \setminus \{0\}$ for the type

$$\sum_{x:V} x \neq 0$$

(c) A *vector bundle* on a type X is a map $V : X \to R\text{-Vect}_n$.

The following defines projective space as the space of lines in a vector space. This is a large type. We will see below, that there is also a small definition of the same type.

Definition 6.1.2 (a) A *line* in a R-vector space V is a subtype $L : V \to \text{Prop}$, such that there exists an $x : V \setminus \{0\}$ with

$$\prod_{y:V} (L(y) \Leftrightarrow \exists c : R.y = c \cdot x)$$

(b) The space of all lines in a fixed n-dimensional vector space V is the *projectivization* of V:

$$\mathbb{P}(V) :\equiv \sum_{L:V \to \text{Prop}} L \text{ is a line}$$

(c) *Projective n-space* $\mathbb{P}^n :\equiv \mathbb{P}(\mathbb{A}^{n+1})$ is the projectivization of $\mathbb{A}^{n+1}$.

Proposition 6.1.3 For any vector space V and line $L \subseteq V$, L is 1-dimensional in the sense that $\|L =_{R\text{-Mod}} R\|$.

Proof Let L be a line. We merely have $x : V \setminus \{0\}$ such that

$$\prod_{y:V} (L(y) \Leftrightarrow \exists c : R.y = c \cdot x)$$

We may replace the "$\exists$" with a "$\sum$", since c is uniquely determined for any x, y. This means we can construct the map $\alpha \mapsto \alpha \cdot x : R \to L$ and it is an equivalence. □

Lines are closed subschemes (at least in $\mathbb{A}^2$):

Proposition 6.1.4 For any line $L : \mathbb{A}^2 \to \mathrm{Prop}$, there merely is a degree one polynomial $P \in R[X_0, X_1]$ such that for all $x : \mathbb{A}^2$, $L(x)$ is equivalent to $P(x) = 0$.

Proof For $x \equiv (a, b)$ a point on L, let P be the polynomial, given by inner product with $(b, -a)$. □

We now give the small construction:

Definition 6.1.5 (using Loc, SQC) Let $n : \mathbb{N}$. *Projective n-space* $\mathbb{P}^n$ is the setquotient of the type $\mathbb{A}^{n+1} \setminus \{0\}$ by the relation

$$x \sim y :\equiv \sum_{\lambda:R} \lambda x = y.$$

By proposition 2.2.3, the non-zero vector y has an invertible entry, so that the right hand side is a proposition and λ is a unit. We write $[x_0 : \cdots : x_n] : \mathbb{P}^n$ for the equivalence class of $(x_0, \ldots, x_n) : \mathbb{A}^{n+1} \setminus \{0\}$.

Theorem 6.1.6 (using SQC, Loc)
$\mathbb{P}^n$ is a scheme.

Proof ...

Proposition 6.1.7 $\mathbb{P}^n$ is separated.

Proof We have to show that $x = y$ is closed for all $x, y : \mathbb{P}^n$. Since we are proving a proposition, we may assume represantatives $[x_0 : \cdots : x_n] = [y_0 : \cdots : y_n]$ and an index i such that x_i is invertible. Let $\lambda :\equiv \frac{y_i}{x_i}$, then $x = y$ is equivalent to

$$\prod_j \lambda x_j = y_j$$

– which is closed. □

Proposition 6.1.8 (using Loc, SQC, Z-choice) $\mathbb{P}^n$ is irreducible.

Proof By proposition 4.8.6 and lemma 4.8.12, $\mathbb{A}^{n+1}$ is irreducible. $\mathbb{A}^{n+1} \setminus \{0\}$ is an open subtype of $\mathbb{A}^{n+1}$, so it is also irreducible by proposition 4.8.11. Finally, the projection $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is surjective, so by lemma 4.8.13, $\mathbb{P}^n$ is irreducible. □

The following conclusion could also be drawn from our results about functions on $\mathbb{P}^n$ in the next section.

Corollary 6.1.9 (using Loc, SQC, Z-choice) $\mathbb{P}^n$ is connected.

Proof Note first that $\mathbb{P}^n$ is pointed by $[1 : 0 : \cdots : 0]$. By proposition 6.1.8, $\mathbb{P}^n$ is irreducible and by proposition 4.8.10 any irreducible pointed type is connected. □

6.2 Functions on $\mathbb{P}^n$

Example 6.2.1 (using Loc) Let $s : \mathbb{P}^1 \to \mathbb{P}^1$ be given by $s([x : y]) :\equiv [x^2 : y^2]$ (see definition 6.1.5 for notation). Let us compute some fibers of s. The fiber $\mathrm{fib}_s([0 : 1])$ is by definition the type

$$\sum_{[x:y]:\mathbb{P}^1} [x^2 : y^2] = [0 : 1].$$

So for any $x : R$ with $x^2 = 0$, $[x : 1] : \mathrm{fib}_s([0 : 1])$ and any other point (x, y) such that $[x : y]$ is in $\mathrm{fib}_s([0 : 1])$, already yields an equivalent point, since y has to be invertible.

This shows that the fiber over $[0 : 1]$ is a first order disk, i.e. $\mathbb{D}(1) = \{x : R | x^2 = 0\}$. The same applies to the point $[1 : 0]$. To analyze $\mathrm{fib}_s([1 : 1])$, let us assume $2 \neq 0$ (in R). Then we know, the two points $[1 : -1]$ and $[1 : 1]$ are in $\mathrm{fib}_s([1 : 1])$ and they are different. It will turn out, that any point in $\mathrm{fib}_s([1 : 1])$ is equal to one of those two. For any $[x' : y'] : \mathrm{fib}_s([1 : 1])$, we can assume $[x' : y'] = [x : 1]$ and $x^2 = 1$, or equivalently $(x - 1)(x + 1) = 0$. By proposition 6.1.7, $\mathbb{P}^1$ is separated and by proposition 4.9.3 this means that inequality is an apartness relation. So for each $x : R$, we know $x - 1$ is invertible or $x + 1$ is invertible. But this means that for any $x : R$ with $(x-1)(x+1) = 0$, that $x = 1$ or $x = -1$.

While the fibers are not the same in general, they are all affine and have the same size in the sense that for each $\operatorname{Spec} A_x :\equiv \operatorname{fib}_s(x)$, we have that A_x is free of rank 2 as an R-module. To see this, let us first note, that $\operatorname{fib}_s([x : y])$ is completely contained in an affine subset of $\mathbb{P}^1$. This is a proposition, so we can use that either x or y is invertible. Let us assume without loss of generality, that y is invertible, then

$$\operatorname{fib}_s([x : y]) = \operatorname{fib}_s([\frac{x}{y} : 1]).$$

The second component of each element in the fiber has to be invertible, so it is contained in an affine subset, which we identify with $\mathbb{A}^1$. Let us rewrite with $z :\equiv \frac{x}{y}$. Then

$$\operatorname{fib}_s([z : 1]) = \sum_{a:\mathbb{A}^1} (a^2 = z) = \operatorname{Spec} R[X]/(X^2 - z)$$

and $R[X]/(X^2 - z)$ is free of rank 2 as an R-module.

Lemma 6.2.2 All functions $\mathbb{P}^1 \to R$ are constant.

Proof ...

Lemma 6.2.3 (using SQC, Loc) Let $p \neq q \in \mathbb{P}^n$ be given. Then there exists a map $f : \mathbb{P}^1 \to \mathbb{P}^n$ such that $f([0 : 1]) = p$, $f([1 : 0]) = q$.

Proof What we want to prove is a proposition, so we can assume chosen $a, b \in \mathbb{A}^{n+1} \setminus \{0\}$ with $p = [a]$, $q = [b]$. Then we set

$$f([x, y]) :\equiv [xa + yb].$$

Let us check that $xa + yb \neq 0$. By ..., we have that x or y is invertible and both a and b have at least one invertible entry. If $xa = -yb$ then it follows that x and y are both invertible and therefore a and b would be linearly equivalent, contradicting the assumption $p \neq q$. Of course f is also well-defined with respect to linear equivalence in the pair (x, y). □

Lemma 6.2.4 Let $n \geq 1$. For every point $p \in \mathbb{P}^n$, we have $p \neq [1 : 0 : 0 : \ldots]$ or $p \neq [0 : 1 : 0 : \ldots]$.

Proof Let $p = [a]$ with $a \in \mathbb{A}^{n+1} \setminus \{0\}$. By ..., there is an $i \in \{0, \ldots, n\}$ with $a_i \neq 0$. If $i = 0$ then $p \neq [0 : 1 : 0 : \ldots]$, if $i \geq 1$ then $p \neq [1 : 0 : 0 : \ldots]$. □

Theorem 6.2.5
All functions $\mathbb{P}^n \to R$ are constant, that is,

$$H^0(\mathbb{P}^n, R) :\equiv (\mathbb{P}^n \to R) = R.$$

Proof ...

6.3 Intersection theory

Example 6.3.1 (using Loc, SQC) It is not the case that for any pair of lines $L, L' \subseteq \mathbb{P}^2$, the R-algebra $R^{L \cap L'}$ is free of rank 1.

Proof The R-algebra $R^{L\cap L'}$ is free of rank 1 if and only if the structure homomorphism $\varphi : R \to R^{L \cap L'}$ is bijective. We will show that it is not even always injective.

Consider the lines

$$L = \{\, [x : y : z] : \mathbb{P}^2 \mid z = 0 \,\}$$

and

$$L' = \{\, [x : y : z] : \mathbb{P}^2 \mid \varepsilon x + \delta y + z = 0 \,\},$$

where ε and δ are elements of R with $\varepsilon^2 = \delta^2 = 0$. Consider the element $\varphi(\epsilon\delta) : R^{L\cap L'}$, which is the constant function $L \cap L' \to R$ with value $\varepsilon\delta$. For any point $[x : y : z] : L \cap L'$, we have $z = 0$ and $\varepsilon x + \delta y = 0$. But also, by definition of $\mathbb{P}^3$, we have $(x, y, z) \neq 0 : R^3$, so one of x, y must be invertible. This implies $\delta \mid \varepsilon$ or $\varepsilon \mid \delta$, and in both cases we can conclude $\varepsilon\delta = 0$. Thus, $\varphi(\epsilon\delta) = 0 : R^{L\cap L'}$.

If φ was always injective then this would imply $\varepsilon\delta = 0$ for any $\varepsilon, \delta : R$ with $\varepsilon^2 = \delta^2 = 0$. In other words, the inclusion

$$\operatorname{Spec} R[X, Y]/(X^2, Y^2, XY) \hookrightarrow \operatorname{Spec} R[X, Y]/(X^2, Y^2)$$

would be a bijection. But the corresponding R-algebra homomorphism is not an isomorphism. □

7 Line bundles

7.1 Line bundles on $\mathbb{A}^1$

Definition 7.1.1 Let X be a type. A *line bundle* is a map $\mathcal{L} : X \to R\text{-Mod}$, such that

$$\prod_{x:X} \|\mathcal{L}_x =_{R\text{-Mod}} R\|.$$

The *trivial line bundle* on X is the line bundle $X \to R\text{-Mod}, x \mapsto R$, and when we say that a line bundle $\mathcal{L}$ is trivial we mean that $\mathcal{L}$ is equal to the trivial line bundle, or equivalently $\|\prod_{x:X} \mathcal{L}_x =_{R\text{-Mod}} R\|$.

Lemma 7.1.2 (using Loc, SQC, Z-choice) For every open subset $U : \mathbb{A}^1 \to \text{Prop}$ of $\mathbb{A}^1$ we have not not: either $U = \emptyset$ or $U = D((X - a_1)\dots(X - a_n)) = \mathbb{A}^1 \setminus \{a_1, \dots, a_n\}$ for pairwise distinct numbers $a_1, \dots, a_n : R$.

Proof For $U = D(f)$, this follows from lemma 3.4.3 because $D(\alpha \cdot (X - a_1)^{e_1} \dots (X - a_n)^{e_n}) = D((X - a_1)\dots(X - a_n))$. In general, we have $U = D(f_1) \cup \dots \cup D(f_n)$ by theorem 4.2.7, so we do not not get (that $U = \emptyset$ or) a list of elements $a_1, \dots, a_n : R$ such that $U = \mathbb{A}^1 \setminus \{a_1, \dots, a_n\}$. Then we can not not get rid of any duplicates in the list. □

Lemma 7.1.3 (using Loc, SQC, Z-choice) Let $U, V : \mathbb{A}^1 \to \text{Prop}$ be two open subsets and let $f : U \cap V \to R^\times$ be a function. Then there do not not exist functions $g : U \to R^\times$ and $h : V \to R^\times$ such that $f(x) = g(x)h(x)$ for all $x : U \cap V$.

Proof By lemma 3.4.3, we can assume

$$\begin{aligned}
U \cup V &= D((X - a_1)\dots(X - a_k)), \\
U &= D((X - a_1)\dots(X - a_k)(X - b_1)\dots(X - b_l)), \\
V &= D((X - a_1)\dots(X - a_k)(X - c_1)\dots(X - c_m)), \\
U \cap V &= D((X - a_1)\dots(X - a_k)(X - b_1)\dots(X - b_l)(X - c_1)\dots(X - c_m)),
\end{aligned}$$

where all linear factors are distinct. Then every function $f : U \cap V \to R^\times$ can by (SQC), lemma 3.4.3 and comparing linear factors not not be written in the form

$$f = \alpha \cdot (X - a_1)^{e_1} \dots (X - a_k)^{e_k} (X - b_1)^{e'_1} \dots (X - b_l)^{e'_l} (X - c_1)^{e''_1} \dots (X - c_m)^{e''_m} \qquad \square$$

with $\alpha : R^\times$, $e_i, e'_i, e''_i : \mathbb{Z}$. Other linear factors can not appear, since they do not represent invertible functions on $U \cap V$. Now we can write $f = gh$ as desired, for example with

$$\begin{aligned}
g &= \alpha \cdot (X - a_1)^{e_1} \dots (X - a_k)^{e_k} (X - b_1)^{e'_1} \dots (X - b_l)^{e'_l}, \\
h &= (X - c_1)^{e''_1} \dots (X - c_m)^{e''_m}.
\end{aligned}$$

Theorem 7.1.4 (using Loc, SQC, Z-choice)
Every $R^\times$-torsor on $\mathbb{A}^1$ (definition 8.3.1) does not not have a global section.

Proof Let T be an $R^\times$-torsor on $\mathbb{A}^1$, that is, for every $x : \mathbb{A}^1$, T_x is a set with a free and transitive $R^\times$ action and $\|T_x\|$. By (Z-choice), we get a cover of $\mathbb{A}^1$ by open subsets $\mathbb{A}^1 = \bigcup_{i=1}^n U_i$ and local sections $s_i : (x : U_i) \to T_x$ of the bundle T. From this we can not not construct a global section by induction on n: Given any two local sections s_i, s_j defined on U_i, U_j, let $f : U_i \cap U_j \to R^\times$ be the unique function with $f(x)s_i(x) = s_j(x)$ for all $x : U_i \cap U_j$. Then by lemma 7.1.3, we not not find $g : U_i \to R^\times$, $h : U_j \to R^\times$ such that the sections $x \mapsto g(x)s_i(x)$ and $x \mapsto h(x)^{-1}s_j(x)$, defined on U_i respectively U_j, agree on $U_i \cap U_j$. This yields a section $\widetilde{s} : (x : U_i \cup U_j) \to T_x$ by lemma 1.2.2 and we can replace U and V by $U \cup V$ in the cover. Finally, when we get to $n = 1$, we have $U_1 = \mathbb{A}^1$ and the global section $s_1 : (x : X) \to T_x$. □

Corollary 7.1.5 (using Loc, SQC, Z-choice) Every line bundle on $\mathbb{A}^1$ is not not trivial.

Proof Given a line bundle L, we can construct an $R^\times$ torsor

$$x \mapsto L_x \setminus \{0\}.$$

Note that there is a well-defined $R^\times$ action on $M \setminus \{0\}$ for every R module M, and the action on $L_x \setminus \{0\}$ is free and transitive and we have $\|L_x \setminus \{0\}\|$ since we merely have $L_x = R$ as R modules. By theorem 7.1.4, there not not is a global section of this torsor, so we have a section $s : (x : \mathbb{A}^1) \to L_x$ with $s(x) \neq 0$ for all $x : \mathbb{A}^1$. But this means that the line bundle L is trivial, since we can build an identification $L_x = R$ by sending $s(x)$ to 1. □

7.2 Regular sections and regular closed subschemes

In classical algebraic geometry, there is the concept of a *generic section* of a line bundle. Informally, the generic sections have the smallest possible vanishing set. The following definition corresponds to this notion:

Definition 7.2.1 Let X be a type and $\mathcal{L} : X \to R$-Mod a line bundle. A section

$$s : \prod_{x:X} \mathcal{L}_x$$

is *regular*, there merely is a trivializing affine cover $U_1 = \operatorname{Spec} A_1, \ldots, U_n = \operatorname{Spec} A_n$ of $\mathcal{L}$, such that each trivialized restriction

$$s_i : \operatorname{Spec} A_i \to R$$

is a regular element (definition 1.3.3) of $(\operatorname{Spec} A_i \to R) = A_i$.

Lemma 7.2.2 Let $s : \operatorname{Spec} A \to R$. s being regular is Zariski-local, i.e. for all Zariski-covers $U_1, \ldots, U_n$ of $\operatorname{Spec} A$, s is regular, if and only if it is regular on all U_i.

Proof It is enough to check this for a localization at $f : A$. Let

$$\frac{s}{1} \cdot \frac{g}{f^k} = 0.$$

then $f^l sg = 0$, which implies $f^l g = 0$ by regularity of s and therefore $\frac{g}{f^l} = 0$. □

Proposition 7.2.3 The choice of trivializing cover in definition 7.2.1 is irrelevant.

Proof By lemma 7.2.2. □

From a line bundle together with a regular section, we can produce a closed subtype of a special kind:

Definition 7.2.4 Let X be a scheme. A *regular closed subtype* of X is a closed subtype $C : X \to \mathrm{Prop}$, such that there merely is an affine open cover $U_1 = \operatorname{Spec} A_1, \ldots, U_n = \operatorname{Spec} A_n$, and $C \cap U_i$ is $V(f_i)$ for a regular $f_i : A_i$.

Lemma 7.2.5 Let $f, g : A$, f be regular and $V(f) = V(g)$, then g is regular and there is a unique unit $\alpha : A^\times$, such that $\alpha f = g$.

Proof $V(f) = V(g)$ implies there are $\alpha, \beta : A$ such that $\alpha f = g$ and $\beta g = f$. But then: $f = \beta g = \beta\alpha f$. So by regularity of f, $\beta\alpha = 1$. By lemma 1.3.4, units are regular and products of regular elements are regular, so g is regular. Uniqueness of α follows from regularity. □

Theorem 7.2.6 (using Z-choice)
Let X be a scheme. For any regular closed subscheme C, there is a line bundle with regular section $(\mathcal{L}, s)$ on X, such that $C = V(s)$.

Proof Let $U_1 = \operatorname{Spec} A_1, \ldots, U_n = \operatorname{Spec} A_n$ be a cover by standard affine opens such that we have regular f_i with $C \cap U_i = V(f_i)$. We define $\mathcal{L}$ to be the trivial line bundle $\_ \mapsto R$ on each U_i and by giving automorphisms on the intersections $U_i \cap U_j :\equiv U_{ij} = \operatorname{Spec} A_{ij}$. On U_{ij}, C is given by $V(\frac{f_i}{1})$ and $V(\frac{f_j}{1})$ which are both regular. Therefore, there is a unit $\alpha : A_{ij}^\times$ such that $\alpha\frac{f_i}{1} = \frac{f_j}{1}$, which we can also view as a map $U_{ij} \to R^\times$ and since $R^\times$ is equivalent to the automorphism group of R as an R-module, this provides the identetification we need to construct $\mathcal{L}$. Under the identification, the local regular sections are identified, so we get a global section s of $\mathcal{L}$, which is locally regular. □

7.3 Line Bundles on $\mathbb{P}^n$

We will construct Serre's twisting sheaves in this section, starting with the "minus first". The following works because of proposition 6.1.3.

Definition 7.3.1 The *tautological bundle* is the line bundle $\mathcal{O}_{\mathbb{P}^n}(-1) : \mathbb{P}^n \to R$-Mod, given by

$$(L : \mathbb{P}^n) \mapsto L.$$

Definition 7.3.2 The *dual* $\mathcal{L}^\vee$ of a line bundle $\mathcal{L} : \mathbb{P}^n \to R$-Mod, is the line bundle given by

$$(x : \mathbb{P}^n) \mapsto \operatorname{Hom}_{R\text{-Mod}}(\mathcal{L}_x, R).$$

8 Bundles and cohomology

In non-synthetic algebraic geometry, the structure sheaf $\mathcal{O}_X$ is part of the data constituting a scheme X. In our internal setting, the scheme X is just a set without any additional data, but when we want to consider the structure sheaf as an object in its own right, then we can represent it by the trivial bundle that assings to every point $x : X$ the set R. Indeed, for an affine scheme $X = \operatorname{Spec} A$, taking the sections of this bundle over a basic open $D(f) \subseteq X$

$$(\prod_{x:D(f)} R) = (D(f) \to R) = A[f^{-1}]$$

yields the localizations of the ring A expected from the structure sheaf $\mathcal{O}_X$. More generally, instead of sheaves of abelian groups, $\mathcal{O}_X$-modules, etc., we will consider bundels of abelian groups, R-modules, etc., in the form of maps from X to the respective type of algebraic structures.

8.1 Quasi-coherent bundles

Sometimes we want to "apply" a bundle to a subtype, like sheaves can be evaluated on open subspaces and introduce the common notation "$M(U)$" for that below. It is, however, not justified to expect, that this application and the corresponding theory of "sheaves" is "the same" as the external one, since the definition below uses the internal hom "$\prod$" – where the corresponding external construction, would be the set of continuous sections of a bundle.

Definition 8.1.1 Let X be a type and $M : X \to R$-Mod a dependent module. Let $U \subseteq X$ be any subtype.

(a) We write:

$$M(U) :\equiv \prod_{x:U} M_x.$$

(b) With pointwise structure, $U \to R$ is an R-algebra and $M(U)$ is a $(U \to R)$-module.

Somewhat surprisingly, localization of modules $M(U)$ can be done pointwise:

Lemma 8.1.2 (using Loc, SQC, Z-choice) Let X be a scheme and $M : X \to R$-Mod a dependent module. For any $f : X \to R$, there is an equality

$$M(X)_f = \prod_{x:X} (M_x)_{f(x)}$$

of $(X \to R)$-modules.

Proof First we construct a map, by realizing that the following is well-defined:

$$\frac{m}{f^k} \mapsto \left(x \mapsto \frac{m(x)}{f(x)^k}\right)$$

So let $\frac{m}{f^k} = \frac{m'}{f^{k'}}$, i.e. let there be an $l : \mathbb{N}$ such that $f^l(mf^{k'} - m'f^k) = 0$. But then we can choose the same $l : \mathbb{N}$ for each $x : X$ and apply the equation to each $x : X$.

We will now show, that the map we defined is an embedding. So let $g, h : M(X)_f$ such that $p : \prod_{x:X} g(x) =_{(M_x)_{f(x)}} h(x)$. Let $m_g, m_h : \prod_{x:X} M_x$ and $k_g, k_h : \mathbb{N}$ such that

$$g = \frac{m_g}{f^{k_g}} \quad \text{and} \quad h = \frac{m_h}{f^{k_h}}.$$

From p we know $\prod_{x:X} \exists_{k_x:\mathbb{N}} f(x)^{k_x}(m_g(x)f(x)^{k_h} - m_h(x)f(x)^{k_g}) = 0$. By proposition 3.3.4, we find one $k : \mathbb{N}$ with

$$\prod_{x:X} f(x)^k(m_g(x)f(x)^{k_h} - m_h(x)f(x)^{k_g}) = 0$$

— which shows $g = h$.

It remains to show that the map is surjective. So let $\varphi : \prod_{x:X}(M_x)_{f(x)}$ and note that

$$\prod_{x:X} \exists_{k_x:\mathbb{N}, m_x:M_x} . \varphi(x) = \frac{m_x}{f(x)^{k_x}}.$$

By proposition 3.3.4 and proposition 4.3.3, we get $k : \mathbb{N}$, an affine open cover $U_1, \dots, U_n$ of X and $m_i : (x : U_i) \to M_x$ such that for each i and $x : U_i$ we have

$$\varphi(x) = \frac{m_i(x)}{f(x)^k}.$$

The problem is now to construct a global $m : (x : X) \to M_x$ from the m_i. We have

$$\prod_{x:U_{ij}} \frac{m_i(x)}{f(x)^k} = \varphi(x) = \frac{m_j(x)}{f(x)^k}$$

meaning there is pointwise an exponent $t_x : \mathbb{N}$, such that $f(x)^{t_x} m_i(x) = f(x)^{t_x} m_j(x)$. By proposition 3.3.4, we can find a single $t : \mathbb{N}$ with this property and define

$$\tilde{m}_i(x) :\equiv f(x)^t m_i(x).$$

Then we have $\tilde{m}_i(x) = \tilde{m}_j(x)$ on all intersections U_{ij}, which is what we need to get a global $m : (x : X) \to M_x$ from lemma 1.2.2. Since $\varphi(x) = \frac{f(x)^t m_i(x)}{f(x)^{t+k}} = \frac{\tilde{m}_i(x)}{f(x)^{t+k}}$ for all i and $x : U_i$, we have found a preimage of φ in $M(X)_f$. □

We will need the following algebraic observation:

Remark 8.1.3 Let M be an R-module and A a finitely presented R-algebra, then there is an R-linear map

$$M \otimes A \to M^{\operatorname{Spec} A}$$

induced by mapping $m \otimes f$ to $x \mapsto x(f) \cdot m$. In particular, for any $f : R$, there is a

$$M_f \to M^{D(f)}.$$

The map $M \otimes A \to M^{\operatorname{Spec} A}$ is natural in M.

Lemma 8.1.4 (using SQC, Loc, Z-choice) Let X be a scheme, $M : X \to R\text{-Mod}$, $U \subseteq X$ open and $f : A$. Then there is an R-linear map

$$M(U)_f \to M(D(f)).$$

Proof Combining lemma 8.1.2 and pointwise application of remark 8.1.3 we get

$$M(U)_f = \left(\prod_{x:U} (M_x)_{f(x)}\right) \to \left(\prod_{x:U} (M_x)^{D(f(x))}\right) = \left(\prod_{x:D(f)} M_x\right) = M(D(f))$$

□

The following definition was used in [Ble17], to characterize quasi coherent sheaves in the little Zariski-topos. We will use it to get a good subcategory of the R-module bundles over a scheme.

Definition 8.1.5 An R-module M is *weakly quasi-coherent*, if for all $f : R$, the canonical homomorphism

$$M_f \to M^{D(f)}$$

from remark 8.1.3 is an equivalence. We denote the type of weakly quasi-coherent R-modules with $R\text{-Mod}_{wqc}$.

Lemma 8.1.6 For any R-linear map $f : M \to N$ of weakly quasi-coherent modules M and N, the kernel of f is weakly quasi-coherent.

Proof Let $K \to M$ be the kernel of f. For any $f : R$, the map $K^{D(f)} \to M^{D(f)}$ is the kernel of $M^{D(f)} \to N^{D(f)}$. The latter map is equal to $M_f \to N_f$ by weak quasi-coherence of M and N and $K_f \to M_f$ is the kernel of $M_f \to N_f$. Let the vertical maps in

$$\begin{array}{ccccc} K_f & \longrightarrow & M_f & \longrightarrow & N_f \\ \downarrow & & \downarrow\simeq & & \downarrow\simeq \\ K^{D(f)} & \longrightarrow & M^{D(f)} & \longrightarrow & N^{D(f)} \end{array}$$

be the canonical maps from remark 8.1.3. The squares commute because of the naturality of the vertical maps. Then the map $K_f \to K^{D(f)}$ is an isomorphism, because by commutativity, it is equal to the induced map between the kernels K_f and $K^{D(f)}$, which has to be an isomorphism, since it is induced by an isomorphism of diagrams. □

Definition 8.1.7 Let X be a scheme. A weakly quasi-coherent bundle on X, is a map $M : X \to R\text{-Mod}_{wqc}$.

An immediate consequence is, that weakly quasi coherent dependent modules have the property that "restricting is the same as localizing":

Lemma 8.1.8 (using SQC, Loc, Z-choice) Let X be a scheme and $M : X \to R$-Mod weakly quasi-coherent, then for all open $U \subseteq X$ and $f : U \to R$ the canonical morphism

$$M(U)_f \to M(D(f))$$

is an equivalence.

Proof By construction of the canonical map from lemma 8.1.4. □

Let us look at an example.

Proposition 8.1.9 Let X be a scheme and $C : X \to R\text{-Alg}_{fp}$. Then C, as a bundle of R-modules, is weakly quasi coherent.

Proof Then for any $f : R$ and $x : X$, using lemma 3.3.1, we have

$$(C_x)_f = C_x \otimes_R R_f = (\operatorname{Spec} R_f \to C_x) = (D(f) \to C_x) = {C_x}^{D(f)}.$$ □

Proposition 8.1.10 (using Loc, SQC, Z-choice) Not every R-module is weakly quasi-coherent in the sense of definition 8.1.5.

Proof We construct a family of R-modules, parametrized by the elements of R, and deduce a contradiction from the assumption that all modules of this family are quasi-coherent.

Given an element $f : R$, the R-module we want to consider is the countable product

$$M(f) :\equiv \prod_{n:\mathbb{N}} R/(f^n).$$

If $f \neq 0$ then $M(f) = 0$ (using proposition 2.2.3). This implies that the R-module $M(f)^{f\neq 0}$ is trivial: any function $f \neq 0 \to M(f)$ can only assign the value 0 to any of the at most one witnesses of $f \neq 0$. If $M(f)$ is quasi-coherent, then this means that $M(f)_f$ is also trivial. Noting that $M(f)$ is not only an R-module but even an R-algebra in a natural way, we have

$$\begin{aligned} M(f)_f = 0 \;&\Leftrightarrow\; \exists k : \mathbb{N}.\; f^k = 0 \text{ in } M(f) \\ &\Leftrightarrow\; \exists k : \mathbb{N}.\; \forall n : \mathbb{N}.\; f^k \in (f^n) \subseteq R \\ &\Leftrightarrow\; \exists k : \mathbb{N}.\; f^k \in (f^{k+1}) \subseteq R. \end{aligned}$$

In summary, if the module $M(f)$ is quasi-coherent for every $f : R$, then the ring R is zero-dimensional in the sense of definition 3.4.1. But this is not the case, as we saw in lemma 3.4.2. □

TODO: $R^{\mathbb{N}}$ is also not weakly quasi-coherent.

Lemma 8.1.11 (using SQC, Loc, Z-choice) Let X be an affine scheme and M_x a weakly quasi-coherent R-module for any $x : X$, then

$$\prod_{x:X} M_x$$

is a weakly quasi-coherent R-module.

Proof We need to show:

$$\left(\prod_{x:X} M_x\right)_f = \left(\prod_{x:X} M_x\right)^{D(f)}$$

for all $f : R$. By weak lemma 8.1.2, quasi-coherence and lemma 8.1.8 we know:

$$\left(\prod_{x:X} M_x\right)_f = \prod_{x:X} (M_x)_{f(x)} = \prod_{x:X} (M_x)^{D(f)} = \left(\prod_{x:X} M_x\right)^{D(f)}.$$

□

Quasi-coherent dependent modules turn out to have very good properties, which are to be expected from what is known about their external counterparts. We will show below, that quasi coherence is preserved by the following constructions:

Definition 8.1.12 Let X, Y be types and $f : X \to Y$ be a map.

(a) For any dependent module $N : Y \to R\text{-Mod}$, the *pullback* or *inverse image* is the dependent module

$$f^*N :\equiv (x : X) \mapsto M_{f(x)}.$$

(b) For any dependent module $M : X \to R\text{-Mod}$, the *push-forward* or *direct image* is the dependent module

$$f_*M :\equiv (y : Y) \mapsto \prod_{x:\mathrm{fib}_f(y)} M_{\pi_1(x)}.$$

Theorem 8.1.13 (using SQC, Loc, Z-choice)
Let X, Y be schemes and $f : X \to Y$ be a map.

(a) For any weakly quasi-coherent dependent module $N : Y \to R\text{-Mod}$, the inverse image f^*N is weakly quasi-coherent.

(b) For any weakly quasi-coherent dependent module $M : X \to R\text{-Mod}$, the direct image f_*M is weakly quasi-coherent.

Proof (a) There is nothing to do, when we use the pointwise definition of weak quasi-coherence.

(b) We need to show, that

$$\prod_{x:\mathrm{fib}_f(y)} M_{\pi_1(x)}$$

is a weakly quasi-coherent R-module. By theorem 5.6.1, the type $\mathrm{fib}_f(y)$ is a scheme. So by lemma 8.1.11, the module in question is weakly quasi-coherent. □

With a non-cyclic forward reference to a cohomological result, there is a short proof of the following:

Proposition 8.1.14 (using SQC, Loc, Z-choice) Let $f : M \to N$ be an R-linear map of weakly quasi-coherent R-modules M and N, then the cokernel N/M is weakly quasi-coherent.

Proof We will first show, that for an R-linear embedding $m : M \to N$ of weakly quasi-coherent R-modules M and N, the cokernel N/M is weakly quasi-coherent. We need to show:

$$(N/M)_f = (N/M)^{D(f)}.$$

By algebra: $(N/M)_f = N_f/M_f$. This means we are done, if $(N/M)^{D(f)} = N^{D(f)}/M^{D(f)}$. To see this holds, let us consider $0 \to M \to N \to N/M \to 0$ as a short exact sequence of dependent modules, over the subtype of the point $D(f) \subseteq 1 = \operatorname{Spec} R$. Then, taking global sections, by theorem 8.3.4, we have an exact sequence

$$0 \to M^{D(f)} \to N^{D(f)} \to (N/M)^{D(f)} \to H^1(D(f), M)$$

– but $D(f) = \operatorname{Spec} R_f$ is affine, so the last term is 0 by theorem 8.3.6 and $(N/M)^{D(f)}$ is the cokernel $N^{D(f)}/M^{D(f)}$.

Now we will show the statement for a general R-linear map $f : M \to N$. By algebra, the cokernel of f is the same as the cokernel of the induced map $M/K \to N$, where K is the kernel of f. By lemma 8.1.6, K is weakly quasi-coherent, so by the proof above, M/K is weakly quasi-coherent. $M/K \to N$ is an embedding, so again by the proof above, its cokernel is weakly quasi-coherent. □

8.2 Finitely presented bundles

We now investigate the relationship between bundles of R-modules on $X = \operatorname{Spec} A$ and A-modules.

Proposition 8.2.1 Let A be a finitely presented R-algebra. There is an adjunction

$$\begin{array}{ccc} M & \longmapsto & (M \otimes x)_{x:\operatorname{Spec} A} \\ A\text{-Mod} & \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} & R\text{-Mod}^{\operatorname{Spec} A} \\ \prod_{x:\operatorname{Spec} A} N_x & \longleftarrow\!\shortmid & N \end{array}$$

between the category of A-modules and the category of bundles of R-modules on $\operatorname{Spec} A$.

For an A-module M, the unit of the adjunction is:

$$\eta_M : M \to \prod_{x:\operatorname{Spec} A} (M \otimes x)$$
$$m \mapsto (m \otimes 1)_{x:\operatorname{Spec} A}$$

Example 8.2.2 (using SQC, Loc) It is not the case that for every finitely presented R-algebra A and every A-module M the map η_M is injective.

Proof Instead of giving a single counterexample, we construct a family of potential counterexamples, indexed by an element $f : R$. We set $A :\equiv R/(f)$ and

$$M :\equiv A^1/\langle\{1 \mid f =_R 0\}\rangle.$$

Then we have $M \otimes x = 0$ for all $x : \operatorname{Spec} A$: an element $x : \operatorname{Spec} A$ is a witness that $f : R$ is invertible and if f is invertible then $A = 0$, so $M = 0$, so $M \otimes x = 0$. This implies that if η_M is injective then $M = 0$. But we have $M = 0$ if and only if $1_A \in \langle\{1 \mid f = 0\}\rangle$ if and only if 1_A a linear combination (of some length n) of elements of the set $\{1 \mid f = 0\}$ if and only if $f = 0$ ($n > 0$) or $1 =_A 0$, that is, f is invertible ($n = 0$). In summary, if η_M is injective for every choice of $f : R$, then every $f : R$ is zero or invertible. But this would be a contradiction to corollary 4.4.4. □

Theorem 8.2.3
Let $X = \operatorname{Spec}(A)$ be affine and let a bundle of finitely presented R-modules $M : X \to R\text{-Mod}_{\mathrm{fp}}$ be given. Then the A-module

$$\tilde{M} := \prod_{x:X} M_x$$

is finitely presented and for any $x : X$ the R-module $\tilde{M} \otimes_A R$ is M_x. Under this correspondence, localizing $\tilde{M}$ at $f : A$ corresponds to restricting M to $D(f)$.

8.3 Cohomology on affine schemes

Definition 8.3.1 Let X be a type and $A : X \to \mathrm{Ab}$ a map to the type of abelian groups. For $x : X$ let T_x be a set with an A_x action.

(a) T is an *A-pseudotorsor*, if the action is free and transitive for all $x : X$.

(b) T is an *A-torsor*, if it is an A-pseudotorsor and

$$\prod_{x:X} \|T_x\|.$$

(c) We write $A\text{-Tors}(X)$ for the type of A-torsors on X.

Torsors on a point are a concrete implementaion of first deloopings:

Definition 8.3.2 Let $n : \mathbb{N}$. A n-th *delooping* of an abelian group A, is a pointed, $(n-1)$-connected, n-truncated type $K(A, n)$, such that $\Omega^n K(A, n) =_{\mathrm{Ab}} A$.

For any abelian group and any n, a delooping $K(A,n)$ exists by [LF14]. Deloopings can be used to represent cohomology groups by mapping spaces. This is usually done in homotopy type theory to study higher inductive types, such as spheres and CW-complexes, but the same approach works for internally representing sheaf cohomology, which is the intent of the following definition:

Definition 8.3.3 Let X be a type and $\mathcal{F} : X \to \mathrm{Ab}$ a dependent abelian group. The k-th cohomology group of X with coefficients in $\mathcal{F}$ is

$$H^k(X,\mathcal{F}) :\equiv \left\| \prod_{x:X} K(\mathcal{F},k) \right\|_0 .$$

Theorem 8.3.4
Let $\mathcal{F},\mathcal{G},\mathcal{H} : X \to \mathrm{Ab}$ be such that for all $x : X$,

$$0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$$

is an exact sequence of abelian groups. Then there is a long exact sequence:

$$\begin{array}{ccccc} & & \dots & \longrightarrow & H^{k-1}(X,\mathcal{H}) \\ & \swarrow & & & \\ H^k(X,\mathcal{F}) & \longrightarrow & H^k(X,\mathcal{G}) & \longrightarrow & H^k(X,\mathcal{H}) \\ & \swarrow & & & \\ H^{k+1}(X,\mathcal{F}) & \longrightarrow & \dots & & \end{array}$$

Proof By applying the long exact homotopy fiber sequence. □

The following is an explicit formulation of the fact, that the Čech-Complex for an $\mathcal{O}_X$-module sheaf on $X = \mathrm{Spec}(A)$ given by an A-module M is exact in degree 1.

Lemma 8.3.5 Let M be a module over a commutative ring A, $F_1,\dots,F_l$ a coprime system on A and for $i,j \in \{1,\dots,l\}$, let $s_{ij} : F_i^{-1}F_j^{-1}M$ such that:

$$s_{jk} - s_{ik} + s_{ij} = 0.$$

Then there are $u_i : F_i^{-1}M$ such that $s_{ij} = u_j - u_i$.

Proof Let $s_{ij} = \frac{m_{ij}}{f_i f_j}$ with $m_{ij} : M$, $f_i : F_i$ and $f_j : F_j$ such that:

$$f_i \cdot m_{jk} - f_j \cdot m_{ik} + f_k \cdot m_{ij} = 0.$$

Let r_i such that $\sum r_i f_i = 1$. Then for

$$u_i :\equiv -\sum_{k=1}^{l} \frac{r_k}{f_i} m_{ik}$$

we have:

$$\begin{aligned} u_j - u_i &= -\sum_{k=1}^{l} \frac{r_k}{f_j} m_{jk} + \sum_{k=1}^{l} \frac{r_k}{f_i} m_{ik} \\ &= -\sum_{k=1}^{l} \frac{r_k}{f_j f_i} f_i m_{jk} + \sum_{k=1}^{l} \frac{r_k}{f_i f_j} f_j m_{ik} \\ &= \sum_{k=1}^{l} \frac{r_k}{f_j f_i}(-f_i m_{jk} + f_j m_{ik}) \\ &= \sum_{k=1}^{l} \frac{r_k}{f_j f_i} f_k m_{ij} \\ &= \frac{m_{ij}}{f_i f_j} \end{aligned}$$

□

Theorem 8.3.6 (using Z-choice, Loc, SQC)
For any affine scheme $X = \mathrm{Spec}(A)$ and coefficients $M : X \to R\text{-Mod}_{wqc}$, we have

$$H^1(X, M) = 0.$$

Proof We need to show, that any M-torsor T on X is merely equal to the trivial torsor M, or equivalently show the existence of a section of T. We have

$$\prod_{x:X} \|T_x\|$$

and therefore, by (Z-choice), there merely are $f_1, \ldots, f_l : A$, such that the $U_i :\equiv \mathrm{Spec}(A_{f_i})$ cover X and there are local sections

$$s_i : \prod_{x:U_i} T_x$$

of T. Our goal is to construct a matching family from the s_i. On intersections, let $t_{ij} :\equiv s_i - s_j$ be the difference, so $t_{ij} : (x : U_i \cap U_j) \to M_x$. By lemma 8.1.8 equivalently, we have $t_{ij} : M(U_i \cap U_j)_{f_i f_j}$. Since the t_{ij} were defined as differences, the condition in lemma 8.3.5 is satisfied and we get $u_i : M(U_i)_{f_i}$, such that $t_{ij} = u_i - u_j$. So we merely have a matching family $\tilde{s}_i :\equiv s_i - u_i$ and therefore, using Lemma 1.2.2 merely a section of T. □

A similar result is provable for $H^2(X, M)$ using the same approach. There is an extension of this result to general n in work in progress [BCW23].

8.4 Čech-Cohomology

In this section, let X be a type, $U_1, \ldots, U_n \subseteq X$ open subtypes that cover X and $\mathcal{F} : X \to \mathrm{Ab}$ a dependent abelian group on X. We start by repeating the classical definition of Čech-Cohomology groups for a given cover.

Definition 8.4.1 (a) For open $U \subseteq X$, we use the notation DUPLICATION

$$\mathcal{F}(U) :\equiv \prod_{x:U} \mathcal{F}_x.$$

(b) For $s : \mathcal{F}(U)$ and open $V \subseteq U$ we use the notation $s :\equiv s_{|V} :\equiv (x : V) \mapsto s_x$.

(c) For a selection of indices $i_1, ..., i_l : \{1, \ldots, n\}$, we use the notation

$$U_{i_1 \ldots i_l} :\equiv U_{i_1} \cap \cdots \cap U_{i_l}.$$

(d) For a list of indices $i_1, \ldots, i_l$, let $i_1, \ldots, \hat{i_t}, \ldots, i_l$ be the same list with the t-th element removed.

(e) For $k : \mathbb{Z}$, the k-th *Čech-boundary operator* is the homomorphism

$$\partial^k : \bigoplus_{i_0, \ldots, i_k} \mathcal{F}(U_{i_0 \ldots i_k}) \to \bigoplus_{i_0, \ldots, i_{k+1}} \mathcal{F}(U_{i_0 \ldots i_{k+1}})$$

given by $\partial^k(s) :\equiv (l_0, \ldots, l_{k+1}) \mapsto \sum_{j=0}^{k} (-1)^j s_{l_0, \ldots, \hat{l_j}, \ldots, l_k | U_{l_0, \ldots, l_{k+1}}}$.

(f) The k-th *Čech-Cohomology group* for the cover $U_1, \ldots, U_n$ with coefficients in $\mathcal{F}$ is

$$\check{H}^k(\{U\}, \mathcal{F}) :\equiv \ker \partial^k / \mathrm{im}(\partial^{k-1}).$$

It is possible to construct a torsor from a čech cocycle:

Lemma 8.4.2 Let A be an abelian group and L a set with decidable equality and $\|L\|$. Let us call $c : (i, j : L) \to A$ a L-cocycle, if $c_{ij} + c_{jk} = c_{ik}$ for all $i, j : L$. Then there is a bijection:

$$\big((T : A\text{-torsor}) \times T^L\big) \to L\text{-cocylces}.$$

Proof Let us first check, that the left side is a set. Let $(T, u), (T', u') : (T : A\text{-torsor}) \times T^L$, then $(T, u) = (T', u')$ is equivalent to $(e : T \cong T') \times ((i : L) \to e(u_i) = e(u'_i))$. But two maps e with this property are equal, since a map between torsors is determined by the image of a single element and L is inhabited.

Assume now $(T, u) : (T : A\text{-torsor}) \times T^L$ to construct the map. Then $c_{ij} :\equiv u_i - u_j$ defines an L-cocycle, because

$$u_i - u_j + u_j - u_k = u_i - u_k.$$

This defines an embedding: Assume (T, u) and (T', u') define the same L-cocycle, then $u_i - u_j = u'_i - u'_j$ for all $i, j : L$. We want to show a proposition, so we can assume there is $i : L$ and use that to get a map $e : T \to T'$ that sends u_i to u'_i. But then we also have

$$e(u_j) = e(u_j - u_i + u_i) = e(u'_j - u'_i + u_i) = u'_j - u'_i + e(u_i) = u'_j - u'_i + u'_i = u'_j$$

for all $j : L$, wich means $(T, u) = (T', u')$.

Now let c be an L-cocylce. Following [Del91], we can define a preimage-candidate:

$$T_c :\equiv \{u : A^L \mid u_i - u_j = c_{ij}\}.$$

A acts on T_c pontwise, since $(a + u_i) - (a + u_j) = u_i - u_j = c_{ij}$ for all $a : A$.

To show that T_c is inhabited, we may assume $i_0 : L$. Then we define $u_i :\equiv -c_{i_0 i}$ to get $u_i - u_j = -c_{i_0 i} + c_{i_0 j} = c_{ij}$.

Now c is of type $(A^L)^L = A^{L \times L}$, so we have an element of the left hand side. Applying the map constructed above yields a cocycle

$$\tilde{c}_{ij} = (k \mapsto c_{ki}) - (k \mapsto c_{kj}) = (k \mapsto c_{ki} - c_{kj}) = (k \mapsto c_{kj} + c_{ji} - c_{kj}) = (k \mapsto c_{ji})$$

– so (T_c, c) is a preimage of c_{ij}. □

Definition 8.4.3 The cover $U_1, \ldots, U_n$ is called *(r-)acyclic* for $\mathcal{F}$, if for all $k : \mathbb{N}$ and $i_0, \ldots, i_k$, we have that the higher (non Čech) cohomology groups are trivial:

$$\forall (r \geq) l > 0. H^l(U_{i_0, \ldots, i_k}, \mathcal{F}) = 0.$$

Example 8.4.4 If X is a scheme, $U_1, \ldots, U_n$ a cover by affine open subtypes and $\mathcal{F}$ pointwise a weakly quasi coherent R-module, then $U_1, \ldots, U_n$ is 1-acyclic for $\mathcal{F}$ by theorem 8.3.6.

Theorem 8.4.5 (using Z-choice)
If $U_1, \ldots, U_n$ is a 1-acyclic cover for $\mathcal{F}$, then

$$\check{H}^1(\{U\}, \mathcal{F}) = H^1(X, \mathcal{F}).$$

Proof Let π be the projection map

$$\pi : \left(\sum_{T : \mathcal{F}\text{-Tors}(X)} \prod_i \prod_{x : U_i} T_x \right) \to \mathcal{F}\text{-Tors}(X).$$

Let us abbreviate the left hand side with $T(\mathcal{F}, U)$. Since the cover is 1-acyclic, π is surjective. With $L_x :\equiv \sum_i U_i(x)$ and lemma 8.4.2 we get:

$$\begin{aligned} T(\mathcal{F}, U) &= \prod_{x : X} \sum_{T_x : \mathcal{F}_x\text{-Tors}} T_x^{L_x} \\ &= \prod_{x : X} L_x\text{-cocycles}. \end{aligned}$$

The latter is the type of čech-1-cocycles (definition 8.4.1 (e)) and in total the equality is given by the isomorphism

$$\iota :\equiv (T, t) \mapsto (i, j \mapsto t_i - t_j) : T(\mathcal{F}, U) \to \ker(\partial^1) \subseteq \bigoplus_{i,j} \mathcal{F}(U_{ij}).$$

Realizing, that $\mathrm{im}(\partial^0)$ corresponds to the subtype of $T(\mathcal{F}, U)$ of trivial torsors, we arrive at the following diagram:

$$
\begin{array}{ccccc}
 & & \mathcal{F}\text{-Tors}(X) & \twoheadrightarrow & H^1(X,\mathcal{F}) \\
 & & \uparrow\!\!\uparrow & & \\
\sum_{T:T(\mathcal{F},U)} \|\pi_1(T) = \mathcal{F}\| & \hookrightarrow & T(\mathcal{F},U) & & \\
\| & & \| & & \\
\operatorname{im}\partial^0 & \hookrightarrow & \ker\partial^1 & \twoheadrightarrow & \check{H}^1(\{U\},\mathcal{F})
\end{array}
$$

The composed map $T(\mathcal{F},U) \to H^1(X,\mathcal{F})$ is a homomorphism and therefore by lemma 1.3.12 a cokernel. So the two cohomology groups are equal, since they are cokernels of the same diagram. □

There is follow-up work in progress [BCW23], that answers the question if Čech cohomology with respect to an acyclic cover agrees with our definition of cohomology in general positively.

9 External justification of axioms

9.1 Justification of Z-choice

Lemma 9.1.1 Let (C,J) be a site, where the Grothendieck topology J is subcanonical. Let

$$f : E \twoheadrightarrow \mathrm{y}(c)$$

be an epimorphism in $\mathrm{Sh}(C,J)$ with representable codomain. Then there is a J-cover $(c_i \to c)_{i\in I}$ of c such that for every i, the pullback of f along $\mathrm{y}(c_i) \to \mathrm{y}(c)$ is a split epimorphism.

$$
\begin{array}{ccc}
E_i & \longrightarrow & E \\
{\scriptstyle f_i}\big\downarrow\!\!\downarrow \;\lrcorner & & \big\downarrow\!\!\downarrow{\scriptstyle f} \\
\mathrm{y}(c_i) & \longrightarrow & \mathrm{y}(c)
\end{array}
$$

Proof By the Yoneda lemma, an epimorphism $E \twoheadrightarrow \mathrm{y}(c)$ is split if and only if the particular element $\mathrm{id}_c \in \mathrm{y}(c)(c)$ is in the image of the map $E(c) \to \mathrm{y}(c)(c)$. Applying the usual characterization of epimorphisms of sheaves [MM12, Corollary III.7.5] to the element $\mathrm{id}_c \in \mathrm{y}(c)(c)$ shows that there is a J-cover $(c_i \xrightarrow{g_i} c)_{i\in I}$ such that for every $i \in I$, there is some $e_i \in E(c_i)$ with $f_{c_i}(e_i) = g_i \in \mathrm{y}(c)(c_i)$. But this means that id_{c_i} is in the image of $(f_i)_{c_i} : E_i(c_i) \to \mathrm{y}(c_i)(c_i)$, as we can see by evaluating the pullback diagram at c_i. So f_i is a split epimorphism. □

Let us formulate a version of the axiom Z-choice in infinitary first-order logic extended with unbounded quantification over objects/sorts ($\exists A.\varphi$, $\forall A.\varphi$) and quantification over functions ($\exists f : A \to B.\varphi$, $\forall f : A \to B.\varphi$) as in Shulmans stack semantics [Shu10, Section 7].

We also use the syntax $\{x : A \mid \varphi(x)\}$ for bounded set comprehension, but this can be translated away. TODO

$$
\begin{aligned}
\varphi_0 &:\equiv \bigwedge_{n,m\in\mathbb{N}} \forall r_1,\dots,r_m : R[X_1,\dots,X_n].\ \varphi_1 \\
\operatorname{Spec} A &:\equiv \{x : R^n \mid \mathrm{ev}(r_1,x) = \cdots = \mathrm{ev}(r_1,x) = 0\} \\
\varphi_1 &:\equiv \forall E.\ \forall \pi : E \to \operatorname{Spec} A.\ ((\forall x : \operatorname{Spec} A.\ \exists e : E.\ \pi(e) = x) \Rightarrow \varphi_2) \\
\varphi_2 &:\equiv \bigvee_{k\in\mathbb{N}} \exists f_1,\dots,f_k : R[X_1,\dots,X_n].\ f_1 + \cdots + f_k = 1 \wedge \varphi_3 \\
D(f_i) &:\equiv \{x : \operatorname{Spec} A \mid \exists y.\ \mathrm{ev}(f_i,x)y = 1\} \\
\varphi_3 &:\equiv \bigwedge_{i=1}^{k} \exists s : D(f_i) \to E.\ \forall x : D(f_i).\ \pi(s(x)) = x
\end{aligned}
$$

10 Type Theoretic justification of axioms

In this section, we present models of the 3 axioms, models that are inspired by type theory. The first model M_1 (Maybe it would be good to give names to the different models?) is a sheaf model, and does not take into account universe (and univalence). This model is built as an *internal* model inside a presheaf model. Rather surprisingly, the same method works if we start from a(n effective) model of univalence. We can localise at a family of open modalities, and obtain in this way a model M_2 of type theory with univalence satisfying the 3 axioms for Zariski topos.

10.1 1-topos model

We first build a 1-sheaf model M_1 of the 3 axioms. This model will be formulated as a model of *dependent type theory*. This will be an *internal* model inside a presheaf model. This is done in two steps, where the first step is a presheaf model on the base category, which is the opposite of the category of finitely presented k-algebras over a fixed commutative ring k. This is a model of (extensional) type theory with a (cumulative) sequence of universes. We define in the second step a property of types, which expresses that this type is a sheaf for Zariski topology. It is direct to show that types satisfying this property are closed by dependent products and dependent sums, and so form a model of dependent type theory. However, the type of sheaves in a given universe is not itself a sheaf[3] and we don't get a model of dependent type theory with universes. We need a refined notion of model in the next subsection to cover this extension.

For any small category $\mathcal{C}$ we can form the presheaf model of type theory over the base category $\mathcal{C}$ [Hof97; Hub16].

A model is given by the following data. First, a collection of contexts Cont, then for two contexts Δ, Γ, a collection of substitutions $\Delta \to \Gamma$, and for each context Γ a collection of types $\mathsf{Type}(\Gamma)$ and finally, for each A in $\mathsf{Type}(\Gamma)$ we have a collection of elements $\mathsf{Elem}(\Gamma, A)$. We also have a collection of operations satisfying some equations, corresponding to the usual operations in dependent type theory, and described in [Hub16].

We describe the model assuming a constructive set theory as our metalanguage. We have a collection of set theoretic universes $\mathcal{U}, \mathcal{V}, \ldots$, and we consider only universes contained in a fixed universe $\mathcal{W}$. We consider next a category $\mathcal{C}$ in $\mathcal{W}$ and describe now the presheaf model over $\mathcal{C}$.

A context is interpreted as a presheaf valued in $\mathcal{W}$ over $\mathcal{C}$. We interpret $\mathsf{Type}(\Gamma)$ as the collection of $\mathcal{W}$-valued presheaves over the category of elements $\int \Gamma$ of Γ, and $\mathsf{Elem}(\Gamma, A)$ as the set of global sections of the presheaf A. The set $\Delta \to \Gamma$ is then the set of natural transformations between the presheaves Δ and Γ. For any set theoretic universe $\mathcal{U}$ (contained in $\mathcal{W}$), we write $\mathsf{Type}_{\mathcal{U}}(\Gamma)$ the set of $\mathcal{U}$-valued presheaves over $\int \Gamma$. Each object X of $\mathcal{C}$ defines a context $Yo(X)$ by the Yoneda embedding. To each set theoretic universe $\mathcal{U}$ strictly contained in $\mathcal{W}$, we can associate a type theoretic universe U by interpreting $U(X)$ to be $\mathsf{Type}_{\mathcal{U}}(Yo(X))$.

We have the truth value presheaf $\Omega_{\mathcal{U}}$ with $\Omega_{\mathcal{U}}(X)$ is the set of $\mathcal{U}$-small valued sieves on X. This type $\Omega_{\mathcal{U}}$ has the usual logical connective (disjunction, conjunction, ...) and existential and universal quantification. We also have an (extensional and strict) equality $F \times F \to \Omega$ for each $\mathcal{U}$-valued presheaf F. We may write simply Ω instead of $\Omega_{\mathcal{U}}$ if $\mathcal{U}$ is not relevant.

We have a dependent type $[p]$ for $p : \Omega$, with $[p]$ subpresheaf of the unit presheaf $\top(X) = \{0\}$. This is a presheaf $[\_]$ over $\int \Omega$, defined by $(A, S) \mapsto \{0 \mid \mathsf{id}_A \in S\}$.

We have a strict truncation operation $F \mapsto \|F\|_S$, $U \to \Omega$, which to any presheaf F on $\int Yo(X)$ associates the sieve of $\alpha : Y \to X$ such that $F(Y, \alpha)$ is inhabited. If $\psi : T \to \Omega$ we then have an equivalence between $\|\Sigma_{x:T}[\psi x]\|_S$ and $\exists_{x:T} \psi x$.

We get in this way a model of type theory with a collection of universes [Hub16] and a type of propositions which is proof-irrelevant and where function extensionality and propositional extensionality are valid.

We look at the special case where $\mathcal{C}$ is the opposite of the category of finitely presented k-algebras for a fixed ring k.

In this model we have a presheaf $R(A) = Hom(k[X], A)$ which has a ring structure.

In the *presheaf* model, we can check that we have $\neg\neg(0 =_R 1)$. Indeed, at any stage A we have a map $\alpha : A \to 0$ to the trivial f.p. algebra 0, and $0 =_R 1$ is valid at the stage 0.

We also have a presheaf Cov, where $Cov(A)$ is the set of finite sequences $f_1, \ldots, f_n \in A$ that are comaximal: 1 is in the ideal generated by $f_1, \ldots, f_n$.

[3] This is exactly this problem that motivated the notion of stacks.

We define $Cov \to \Omega,\ c \mapsto p_c$ by defining p_c to be the proposition $\mathrm{inv}(f_1) \vee \cdots \vee \mathrm{inv}(f_n)$: $p_c(A)$ is the sieve of $\alpha : A \to B$ such that one $\alpha(f_i)$ is invertible in B. In particular, for $n = 0$, this is the proposition $\bot$ (empty presheaf).

We can then define the property $\mathsf{isSheaf}$ of being a sheaf for a type T as being the fact that the diagonal maps $\delta_c : T \to T^{[p_c]}$ are isomorphisms for all $c : Cov$. If $f : T \to T'$ is a map, we can define $\mathsf{isIso} f$ which expresses that f is a bijection, in the sense of the present (extensional) presheaf model of type theory. The property $\mathsf{isSheaf}$ can then be defined as $\mathsf{isSheaf} = \forall_{c:Cov} \mathsf{isIso}\delta_c$ and is of type $U \to \Omega$.

Any representable presheaf is a sheaf by algebraic facts. If B is a finitely presented k-algebra to check that $Yo(B)$ is a sheaf reduces to a local-global principle: that for any covering $f_1, \dots, f_n$ of a finitely presented algebra A, given a compatible family of elements $u_i : A_{f_i}$ there exists a unique element u in A such that $\alpha_i u = u_i$ for the localisation map $\alpha_i : A \to A_{f_i}$.

One can check directly that $\Pi_{x:A} B$ is a sheaf if B is a family of sheaves over A and that $\Sigma_{x:A} B$ is a sheaf if A is a sheaf and B is a family of sheaves. We hence get a model of type theory by interpreting a type as a sheaf. A context is still interpreted as a presheaf, while a type is now a dependent presheaf T with a proof of $\mathsf{isSheaf}(T)$. An element is a section of the underlying dependent presheaf. We have a new type of propositions Ω_s which is the subpresheaf of Ω of sieves satisfying the sheaf condition. We can check that Ω_s itself satisfies the sheaf condition. If F is a sheaf, then $a =_F b$ is in Ω_s for a and b in F. We have a modal operation $M : \Omega \to \Omega_s$, and the logical operations $\bot, \vee, \exists$ are redefined using this modal operation.

If we have two maps $\beta : A \to B$ and $\gamma : A \to C$ we define $\beta \otimes \gamma : A \to B \otimes_A C$ which is a coproduct in the category of maps of domain A. Any map $\alpha : A \to B$ defines a f.p. algebra B_α over R defined at stage A, with $B_\alpha(C, \gamma) = B \otimes_A C$ and $Sp(B_\alpha)$ is the (pre)sheaf defined at stage A represented by α. If F is a presheaf defined at stage A, we have $F^{Sp(B_\alpha)}(C, \gamma) = F(B \otimes_A C, \beta \otimes \gamma)$. In particular, we have

$$R^{Sp(B_\alpha)}(C, \gamma) = R(B \otimes_A C, \beta \otimes \gamma) = B \otimes_A C = B_\alpha(C, \gamma)$$

which shows that the second axiom is justifed in our model.

In this new model R is a local ring. In particular, we have $\neg(0 =_R 1)$, since $M \bot$ is $0 =_R 1$.

However the type of sheaves in a given universe $\Sigma_{X:\mathcal{U}}[\mathsf{isSheaf}(X)]$ is not a sheaf.

We shall present another model which both interprets a cumulative hierarchy of universes(s?) and *univalence* and *propositional truncation*.

10.2 Some properties of the sheaf model

10.2.1 Quasi-compact propositions

We recall [LQ15] that the Zariski lattice $Z(A)$ of a ring A is defined as the lattice generated by symbols $D(a),\ a \in A$ with relations

$$D(0) = 0 \qquad D(1) = 1 \qquad D(ab) = D(a) \wedge D(b) \qquad D(a + b) \leqslant D(a) \vee D(b)$$

The following can be proved directly.

Proposition 10.2.1 *Z defines a sheaf and is internally a sublattice of Ω, which corresponds to the sublattice of quasi-compact propositions.*

As noted above, a map $\alpha : A \to B$ defines a f.p. algebra B_α over R defined at stage A, with $B_\alpha(C, \gamma) = B \otimes_A C$ and $Sp(B_\alpha)$ is the (pre)sheaf defined at stage A represented by α. We can then check

$$Z(R)^{Sp(B_\alpha)}(C, \gamma) = Z(R)(B \otimes_A C, \beta \otimes \gamma) = Z(B \otimes_A C) = Z(B_\alpha)(C, \gamma)$$

which shows that the equality $Z(R)^{Sp(D)} = Z(D)$ is valid in the sheaf model for any f.p. R-algebra D.

10.2.2 Quasi-coherence

A module M in the sheaf model defined at stage A, where A is a f.p. k-algebra, is given by a sheaf over the category of elements of A. It is thus given by a family of modules $M(B, \alpha)$, for $\alpha : A \to B$, and restriction maps $M(B, \alpha) \to M(C, \gamma\alpha)$ for $\gamma : B \to C$. In general this family is not determined by its value $M_A = M(A, \mathsf{id}_A)$ at A, id_A.

Proposition 10.2.2 *M is internally quasi-coherent iff we have $M(B, \alpha) = M_A \otimes_A B$ and the restriction map for $\gamma : B \to C$ is $M_A \otimes_A \gamma$.*

10.2.3 Projective space

We have defined $\mathbb{P}^n$ to be the set of lines in $V = R^{n+1}$, so we have

$$\mathbb{P}^n \;=\; \Sigma_{L:V\to\Omega}[\exists_{v:V}\neg(v=0)\wedge L = Rv]$$

The following was noticed in [KR77].

Proposition 10.2.3 $\mathbb{P}^n(A)$ is the set of submodules of A^{n+1} factor direct in A^{n+1} and of rank 1.

We recall [LQ15] that a vector $u = (u_0, \ldots, u_n)$ in A^{n+1} is unimodular iff the ideal generated by $u_0, \ldots, u_n$ in A contains 1.

Proof $\mathbb{P}^n$ is the set of pairs $L, 0$ where $L : \Omega^V(A)$ satisfies the proposition $\exists_{v:V}\neg(v = 0) \wedge L = Rv$ at stage A. This condition implies that L is a quasicoherent submodule of R^{n+1} defined at stage A. It is thus determined by its value $L(A, \mathsf{id}_A) = L_A$.

Furthermore, the condition also implies that L_A is locally free of rank 1. By local-global principle [LQ15], L_A is finitely generated. We can then apply Theorem 5.14 of [LQ15] to deduce that L_A is factor direct in A^{n+1} and of rank 1. □

One point in this argument was to notice that the condition

$$\exists_{v:V}\neg(v = 0) \wedge L = Rv$$

implies that L is quasi-coherent. This would be direct in presence of univalence, since we would have then $L = R$ as a R-module and R is quasi-coherent. But it can also be proved without univalence by transport along isomorphism: a R-module which is isomorphic to a quasi-coherent module is itself quasi-coherent.

10.3 Models of univalence

The constructive models of univalence are presheaf models parametrised by an interval object $\mathbf{I}$ (presheaf with two global distinct elements 0 and 1 and which is tiny) and a classifier object Φ for cofibrations. The model is then obtained as an internal model of type theory inside the presheaf model. For this, we define $C : U \to U$, uniform in the universe U, which is closed by dependent product and sum, and which satisfies, for $A : U^{\mathbf{I}}$, the transport principle

$$(\Pi_{i:\mathbf{I}} C(Ai)) \to (A0 \to A1)$$

We get then a model of univalence by interpreting a type as a presheaf A together with an element of $C(A)$.

This is over a base category $\mathcal{B}$.

If we have another category $\mathcal{C}$, we automatically get a new model of univalent type theory by changing $\mathcal{B}$ to $\mathcal{B} \times \mathcal{C}$.

A particular case is if $\mathcal{C}$ is the opposite of the category of f.p. k-algebras, where k is a fixed commutative ring.

We have the presheaf R defined by $R(J, A) = Hom(k[X], A)$ where J object of $\mathcal{B}$ and A object of $\mathcal{C}$.

The presheaf $\mathsf{G_m}$ is defined by $\mathsf{G_m}(J, A) = Hom(k[X, 1/X], A) = A^\times$, the set of invertible elements of A.

10.4 Propositional truncation

We start by giving a simpler interpretation of propositional truncation. This will simplify the proof of the validity of Zariski local choice in the model.

We work in the presheaf model over a base category $\mathcal{B}$ which interprets univalent type theory, with a presheaf Φ of cofibrations. The interpretation of the propositional truncation $\|T\|$ *does not* require the use of the interval $\mathbf{I}$.

We recall that in the models, to be contractible can be formulated as having an operation $\mathsf{ext}(\psi, v)$ which extends any partial element v of extent ψ to a total element.

The (new) remark is then that to be a (h)proposition can be formulated as having instead an operation $\mathsf{ext}(u, \psi, v)$ which, now *given* an element u, extends any partial element v of extent ψ to a total element.

Propositional truncation is defined as follows. An element of $\|T\|$ is either of the form $\mathsf{inc}(a)$ with a in T, or of the form $\mathsf{ext}(u, \psi, v)$ where u is in $\|T\|$ and ψ in Φ and v a partial element of extent ψ.

In this definition, the special constructor ext is a "constructor with restrictions" which satisfies $\mathsf{ext}(u, \psi, v) = v$ on the extent ψ.

10.5 Covering

We want to force $e_c \coloneqq \|\mathrm{inv}(f_1) + \cdots + \mathrm{inv}(f_n)\|$ for $f_1, \ldots, f_n$ in Cov.

Note that we have $e_c \to p_c$, with p_c is the *strict* proposition $\mathrm{inv}(f_1) \vee \cdots \vee \mathrm{inv}(f_n)$. If we work in an effective metatheory, without choices, we cannot expect to have $p_c \to e_c$.

Lemma 10.5.1 The diagonal map $\sigma : R \to R^{e_c}$ is an isomorphism.

Proof It is enough to define a map $\mathsf{patch} : R^{e_c} \to R$ such that $\mathsf{patch}(\sigma(r)) = r$ since e_c is a hproposition.

If we are at a stage A and $f_1, \ldots, f_n$ comaximal in A, and $u : R^{e_c}(A)$ we define $\mathsf{patch}(u)$ as follows. Note that e_c becomes inhabited in each $A[1/f_i]$. We get then a family of elements u_i in each $A[1/f_i]$, with u_i compatible with u_j. We can then patch these elements together to an element in A. ☐

(Since R is a strict hset, the canonical map $R^{p_c} \to R^{e_c}$ is an isomorphism.)

We now consider the model which, intuitively, forces all these hpropositions e_c to be contractible. This is obtained by restricting ourselves to *modal types* T, the types T such that the diagonal map $T \to T^{e_c}$ is an *equivalence*.

We have just seen that the strict hset R is such a modal type.

The same reasoning actually shows that each representable presheaf, in particular $\mathsf{G_m}$, is a modal (strict) hset.

We get a model of type theory with univalence where a type is interpreted as a modal type. In this model $\bot$ is interpreted as $1 = 0$. We can check that if T is modal then $T^{1=0}$ is contractible (using the covering with $n = 0$), since this is equivalent to $(T^{\bot})^{1=0}$.

If P is a proposition, then P is modal if, and only if, we have $P^{e_c} \to P$. (Thus $\bot$ is not modal, since we don't have $\neg\neg(1 = 0)$; indeed we have instead $\neg(1 = 0)$ in the original presheaf model.) Since P is a proposition, P^{e_c} is equivalent to $P^{\mathrm{inv}(f_1)} \times \cdots \times P^{\mathrm{inv}(f_n)}$.

We can use this characterisation of modal proposition to define the interpretation of $\|T\|(A)$ in the *sheaf* model.

An element in $\|T\|(A)$ is

1. either $\mathsf{inc}(a)$ with a in $T(A)$,
2. or of the form $\mathsf{ext}(u, \psi, v)$ with u in $\|T\|(A)$ and ψ in Φ and v in $\|T\|(A)$ of extent ψ,
3. or of the form $\mathsf{cov}(f_1, u_1, \ldots, f_n, u_n)$ with $f_1, \ldots, f_n$ in $Cov(A)$ and u_i in $\|T\|(A[1/f_i])$.

If $\alpha : A \to B$ we define the restriction operation inductively as follows.

1. $\mathsf{inc}(a)|\alpha = \mathsf{inc}(a|\alpha)$
2. $\mathsf{ext}(u, \psi, v)|\alpha = \mathsf{ext}(u|\alpha, \psi, v|\alpha)$
3. $\mathsf{cov}(f_1, u_1, \ldots, f_n, u_n)|\alpha = \mathsf{cov}(\alpha f_1, u_1|\alpha_1, \ldots, \alpha f_n, u_n|\alpha_n)$ where α_i is the composition $A \to B \to B[1/\alpha f_i]$.

If P is a modal proposition and $u : T \to P$ we can define a map $v : \|T\| \to P$ such that $v \circ \mathsf{inc} = u$ as a strict equality.

We can then define $l : \Pi_{f_1,\ldots,f_n:R} \|\mathrm{inv}(f_1) + \cdots + \mathrm{inv}(f_n)\|$.

10.6 Zariski local choice

If $c = f_1, \ldots, f_n$ is a covering of A and $P : Sp(A) \to \mathcal{U}$ we define $\Pi_c P$ to be $\Pi_{D(f_1)} P \times \Pi_{D(f_n)} P$. In this way, Π_c is an operation $\mathcal{U}^{Sp(A)} \to \mathcal{U}$.

We prove local choice: if A is a f.p. algebra over R then we have a map

$$l : (\Pi_{x:Sp(A)} \|P\|) \to \left\|\Sigma_{c:Cov(A)} \Pi_c T\right\|$$

At a stage B a f.p. algebra over R is given by $B \to A$ and we have $Yo(B).Sp(A)$ isomorphic to $Yo(A)$.

For defining the map l, we define $l(v)$ by induction on v. The element v is in $(\Pi_{x:Sp(A)} \|P\|)(B)$, which can be seen as an element of $\|P\|(A)$. If it is $\mathsf{inc}(u)$ we associate $\mathsf{inc}(1, u)$ with the covering 1 of A. If it is $\mathsf{ext}(u, \psi, v)$ the image is $\mathsf{ext}(l(u), \psi, l(v))$. If it is $\mathsf{cov}(f_1, u_1, \ldots, f_n, u_n)$ we have by induction $l(u_i)$ in $\left\|\Sigma_{c:Cov(A[1/f_i])} \Pi_c P\right\|$. We can then conclude using the law $\|A\| \times \|B\| \to \|A \times B\|$.

The fact that R is local holds like in the 1-topos case, and similarly for the fact that $A \to R^{Sp(A)}$ is an isomorphism for A f.p. over R.

This model is a simplified version of the sheaf model considered in [CRS21]. It contains however the expected notion of descent data. The following Lemma illustrates what is going on. If p is a proposition, a partial element of a type T of extent p is an element of T^p.

Lemma 10.6.1 Let p_0, p_1, p_2 be propositions. The type $T^{p_0 \vee p_1 \vee p_2}$ is equivalent to the type of tuples u_0, u_1, u_2 where u_i is a partial element of extent p_i together with paths $u_{ij} : u_i =_T u_j$ of extent $p_i \wedge p_j$ satisfying the cocycle condition $u_{01} \cdot u_{12} = u_{02}$ on the extend $p_0 \wedge p_1 \wedge p_2$.

We can generalize this as follows. If $c = f_1, \ldots, f_n : Cov(A)$ and T is a presheaf defined at level A, we define $D_c(T)(A)$, a descent data for T for the covering c, as the type of family $u_K(i) : T[1/f_K]$ where K is a nonempty finite subset of $1, \ldots, n$ and i an element of $\mathbf{I}^K$ such that $\vee_p(i(p) = 1)$ and $u_K(i) = u_L(i_{|L})$ if $K = L, p$ and $i(p) = 0$. We can then check that $D_c(T)$ is equivalent to $T^{inv(f_1) \vee \cdots \vee inv(f_n)}$. So T is a sheaf iff the canonical map $T \to D_c(T)$ is an equivalence.

If T is a hset (0-type), we recover the usual patching condition: if we have $u_i : T[1/f_i]$ with an equality $u_i = u_j$ on $T[1/f_i f_j]$ we can find u in $T(A)$ such that $u = u_i$ on $T[1/f_i]$.

If T is a 1-type, we recover the usual patching condition: if we have $u_i : T[1/f_i]$ with an equality $e_{ij} : u_i = u_j$ on $T[1/f_i f_j]$, with the cocycle condition $e_{ij} \cdot e_{jk} = e_{ik}$, we can find u in $T(A)$ with $e_i : u = u_i$ on $T[1/f_i]$ such that $e_{ij} \cdot e_j = e_i$.

10.7 Global sections and Zariski global choice

If T is a sheaf defined at level A, we let $\square_A T$ the type of global sections. If $c = f_1, \ldots, f_n$ is in $Cov(A)$ we let $\square_c T$ be the type $\square_{A[1/f_1]} T \times \cdots \times \square_{A[1/f_n]} T$.

Using these notations, we can state the principle of Zariski global choice

$$(\square \, \|T\|) \leftrightarrow \left\| \Sigma_{c:Cov(k)} \square_c T \right\|$$

This is valid in the present model.

Using this principle, we can show that $\square K(\mathsf{G_m}, 1)$ is equal to the type of projective modules of rank 1 over k and that each $\square K(R, n)$ for $n > 0$ is contractible.

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