Synthetic Finite Schemes

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Abstract

These are notes on work in progress on finite schemes in synthetic algebraic geometry.

1 Definition of finite schemes

We use definitions and results from [CCH23] and [Che+23].

There are a couple of equivalent definitions of finite schemes, which we will introduce and show to be equivalent in this section.

Definition 1.1 A type X is a *finite scheme* if it is of the form $X = \operatorname{Spec} A$ for a finitely presented R-algebra which is finitely generated as an R-module.

Example 1.2 (a) Finite types

(b) Infinitesimal disks $\mathbb{D}_k(n) \subseteq \mathbb{A}^n$ of order k

(c) Closed propositions

Theorem 1.3
Let $X = \operatorname{Spec}(A)$ be an affine scheme, then the following are equivalent:

(i) X is a finite scheme.

(ii) A is a finitely presented R-module.

(iii) WIP: There are finitely presented R-algebras B with a surjective homomorphism $B \to A$, A is a finitely presented R-module and B is finite free as an R-module.

(iv) X is projective.

(v) X is compact.

Proof (i) ⇔ (iii): By constructive reading of Tag 0564 in the Stacks Project (TODO: turn into reference). For generators e_i of A, B is defined as $R[X_1, \ldots, X_n]/(P_1, \ldots, P_n)$ where P_i are monic polynomials such that $P_i(e_i) = 0$ in A.

(ii) ⇒ (iv): (The 'same proof' without the 'finite free' assumption should work.) Let A be finite free for now. We consider the projective space $\mathbb{P}A^\star$ associated with the R-linear dual of A. This is $\mathbb{P}^{n-1}$ after choosing a basis of A. Given $[\varphi] : \mathbb{P}A^\star$ we consider the proposition $C([\varphi])$ that $\varphi(1)\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y : A$. This is well-defined and a closed proposition because it suffices to check it for basis elements of A. $C([\varphi])$ implies $\varphi(1) \neq 0$ because otherwise $\varphi(x)^2 = 0$ for all $x : A$ and then φ is not-not zero (projective space contains only non-zero vectors). So $x \mapsto \varphi(x)/\varphi(1)$ determines a point of Spec A for $[\varphi] : \mathbb{P}A^\star$ such that $C([\varphi])$ holds (and one can go in the reverse direction, and verify that the two maps are inverse to each other).

(iv) ⇒ (v): Projective schemes are compact by [Che+23, Theorem 3.0.7].

(v) ⇒ (i): (TODO: copy from #6) □

Lemma 1.4 Finite schemes are closed under dependent sums and identity types.

Proof Compact types and affine types are both closed under dependent sums ([Che+23, Lemma 2.0.3]), so by the characterization in Theorem 1.3, finite schemes are closed under dependent sums. Finite schemes are affine, so their identity types are closed propositions, which are finite schemes. □

It is possible to prove that finite schemes are compact without using the compactness of $\mathbb{P}^n$:

Proposition 1.5 Let A be a finitely presented R-algebra. If furthermore A is finitely generated as an R-module, then $X = \operatorname{Spec}(A)$ is compact (in the sense that X-indexed products of opens are open).

Proof Let $A = R[X_1, \ldots, X_k]/(q_1, \ldots, q_t)$. As A is finitely generated as an R-module, there are monic polynomials $f_1, \ldots, f_k$ of positive degree such that $f_\ell(X_\ell) = 0$ in A. Hence $\operatorname{Spec}(A)$ is a closed subset of $\prod_{\ell=1}^{k} \operatorname{Spec}(R[X_\ell]/(f_\ell))$. As closed subsets of compact sets are compact ([Che+23, Lemma 2.0.3]+closed propositions are compact) and finite products of compact sets are compact ([Che+23, Lemma 2.0.3]), we are reduced to the situation that $A = R[X]/(f)$ where $f = \sum_{j=0}^{n} a_{n-j} X^j$ is a monic polynomial of positive degree n. In this case X is the set of zeros of f and it suffices to prove: For every finite list $g_1, \ldots, g_m : R[X]$ of polynomials, the proposition that

$$\forall (u : R).\ \big(f(u) = 0 \Rightarrow \bigvee_{i=1}^{m} g_i(u) \neq 0\big) \tag{†}$$

is open. To this end, we consider the polynomial

$$p(U_1, \ldots, U_n, T) := \prod_{j=1}^{n} \sum_{i=1}^{m} g_i(U_j) T^{i-1}.$$

Regarded as a polynomial in T, its coefficients are symmetric in the U_i. By the fundamental theorem on symmetric polynomials, there are polynomials $h_0, \ldots, h_m : R[A_0, \ldots, A_{n-1}]$ such that

$$p(U_1, \ldots, U_n, T) = \sum_{i=1}^{m} h_i(e_1(\vec{U}), \ldots, e_n(\vec{U})) T^{i-1}.$$

We claim that proposition (†) is equivalent to the disjunction

$$\bigvee_{i=1}^{m} (h_i(a_1, \ldots, a_n) \neq 0). \tag{‡}$$

Assume Proposition (†). As Proposition (‡) is negative and hence double negation stable, we may assume that f splits into linear factors: $f(X) = \prod_{j=1}^{n}(X - u_j)$. By assumption, for every $j \in \{1, \ldots, n\}$ we have $\bigvee_{i=1}^{m}(g_i(u_j) \neq 0)$. Hence

$$1 \in \bigcap_{j=1}^{n} \big(g_i(u_j)\big)_{i=1}^{m} = c\Big(\sum_{i=1}^{m} g_i(u_j) T^{i-1}\Big) = c(p) = \big(h_i(a_1, \ldots, a_n)\big)_{i=1}^{m}, \tag{$\star$}$$

so Proposition (‡) holds. Here c refers to the radical content of a polynomial, the radical of the ideal generated by its coefficients, and the second equality is by [**banaschewski-vermeulen:radical**].

Conversely, assume Proposition (‡) and let $u : R$ be a zero of f. As the claim that $\bigvee_{i=1}^{m}(g_i(u) \neq 0)$ is double negation stable, we may assume that f splits into linear factors, $f(X) = \prod_{j=1}^{n}(X - u_j)$, with $u_1 = u$. By ($\star$), we have $1 \in \big(g_i(u_1)\big)_{i=1}^{m}$ as desired. ☐

2 Quasi-finite schemes

Definition 2.1 A proposition p holds *foo-locally* if and only if there are numbers $a_1, \ldots, a_n : R$ such that for every partition $\{1, \ldots, n\} = I \mathbin{\dot\cup} J$, if all the a_i with $i \in I$ are zero and all the a_j with $j \in J$ are invertible, then p holds.

Definition 2.2 A scheme is *quasi-finite* if and only if foo-locally, it is finite.

Example 2.3 (a) Finite schemes are quasi-finite.

(b) Open propositions are quasi-finite.

Proposition 2.4 Finite schemes are quasi-finite and compact.

Proof Compactness is by Proposition 1.5. ☐

XXX Question: Does the converse hold? Classically it is well-known. Need to check issue #6.

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References

[CCH23] Felix Cherubini, Thierry Coquand, and Matthias Hutzler. *A Foundation for Synthetic Algebraic Geometry*. 2023. arXiv: 2307.00073 [math.AG]. URL: https://www.felix-cherubini.de/iag.pdf (cit. on p. 1).

[Che+23] Felix Cherubini et al. *Proper Synthetic Schemes*. 2023. URL: https://www.felix-cherubini.de/proper.pdf (cit. on pp. 1, 2).