Exercises

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This is the place for exercises in synthetic algebraic geometry.

1 Nullstellensatz

Definition 1.1 Let A be an R-algebra, then we define the nilradical $\mathrm{Nil}(A)$ of A as the ideal of nilpotents in A.

Definition 1.2 Let A be an R-algebra, then we define the Jacobson radical $\mathrm{Jac}(A)$ of A as the ideal of $a : A$ such that forall $b : A$ we have $1 - ba$ invertible.

Exercise 1 Prove that for any R-algebra A we have:

$$\mathrm{Nil}(A) \subset \mathrm{Jac}(A)$$

Exercise 2 Prove that we have:

$$\mathrm{Nil}(R) = \mathrm{Jac}(R)$$

Hint: Remember $x \neq 0$ if and only if x is invertible, and $\neg\neg(x = 0)$ if and only if x is nilpotent.

Exercise 3 Prove that for any f.p. algebra A and $a : A$, we have that:

(i) a is nilpotent if and only if $a(x)$ is nilpotent for all $x : \mathrm{Spec}(A)$.

(ii) a is invertible if and only if $a(x)$ is invertible for all $x : \mathrm{Spec}(A)$.

Exercise 4 Prove that for any f.p. algebra A, we have that:

$$\mathrm{Nil}(A) = \mathrm{Jac}(A)$$

Hint: Use exercises 2 and 3.

2 Closed dense subtypes

Exercise 1 Let I be a f.g. ideal and let P be the closed proposition $I = 0$. Show that we have $\neg\neg P$ if and only if I is nilpotent.

Definition 2.1 For any type X, a subtype $P \subset X$ is called dense if for any open subtype $U \subset X$ we have that $U \cap P = \emptyset$ implies $U = \emptyset$

Recall that giving a subtype $P \subset X$ is equivalent to giving a map $X \to \mathrm{Prop}$

Exercise 2 Let X be any type with a closed subtype $C : X \to \mathrm{Closed}$. Show that $C \subset X$ is dense if and only:

$$\prod_{x:X} \neg\neg C(x)$$

Exercise 3 Assume that A is a f.p. algebra and I a f.g. A-ideal. Show that the subscheme:

$$\mathrm{Spec}(A/I) \subset \mathrm{Spec}(A)$$

is closed. Show that it is dense if and only if I is nilpotent.

3 The Zariski lattice

Exercise 1 Show that the canonical map $A[X] \to R[X]^{Sp(A)}$ is an isomorphim

Here are two applications:

Exercise 2: Zariski lattice If A is a ring, one defines (Joyal) the Zariski lattice $Zar(A)$ as the distributive lattice generated by symbols $D(a)$ for a in A and relations:

$$\begin{aligned} D(0) &= 0 \\ D(1) &= 1 \\ D(a+b) &\leq D(a) \vee D(b) \\ D(ab) &= D(a) \wedge D(b) \end{aligned}$$

(This can be realized as the lattice of radicals of finitely generated ideals.)

Prove that $Zar(R)$, where R is the generic local ring, is the set of open propositions, with $D(r)$ being $r \neq 0$.

If A is a finitely presented R-algebra, prove that the canonical map $Zar(A) \to Zar(R)^{Sp(A)}$ is an isomorphism. Hint: use the surjective map $R[X] \to Zar(R)$, $\Sigma r_i X^i \mapsto D(r_0, \ldots, r_n)$ and Zariski local choice.

Definition 3.1 Let A be a ring, $e_0, \cdots, e_n$ is a *fundamental system of idempotents* in A if:

$$\begin{aligned} e_i^2 &= e_i \\ e_i e_j &= 0 \text{ if } i \neq j \\ e_0 + \cdots + e_n &= 1 \end{aligned}$$

Exercise 3 Given a fundamental system of idempotents $e_0, \cdots, e_n$ in a f.p. algebra A, we have that:

$$\mathrm{Spec}(A) = D(e_0) + \cdots + D(e_n)$$

Exercise 4 Given a function $f : Sp(A) \to \mathbb{N}$, show that there exists a fundamental system of idempotents $e_0, \cdots, e_n$ in A such that for all $x : \mathrm{Spec}(A)$ we have that $x \in D(e_i)$ if and only if $f(x) = i$.

Hint: Look at the map $Sp(A) \to R[X]$, $x \mapsto X^{f(x)}$, the corresponding polynomial in $A[X]$ will be $\Sigma_i e_i X^i$.

4 Functions on projective spaces

Exercise 1 Show that any map in:

$$\mathbb{P}^1 \to R$$

is constant.

Hint: Present $\mathbb{P}^1$ as a pushout $\mathbb{A}^1 \coprod_{\mathbb{A}^\times} \mathbb{A}^1$, then compute $R^{\mathbb{P}^1}$ as a pullback.

Exercise 2 Assume given $x, y : \mathbb{P}^n$ such that $x \neq y$. Prove there is a map:

$$\psi : \mathbb{P}^1 \to \mathbb{P}^n$$

such that $\psi([1:0]) = x$ and $\psi([0:1]) = y$.

Exercise 3 Show that any map in:

$$\mathbb{P}^n \to R$$

is constant. Deduce that $\mathbb{P}^n$ is not affine for any $n > 0$.

Hint: You can prove that for any $x : \mathbb{P}^n$ we have that $x \neq [1:0:0:\cdots:0]$ or $x \neq [0:1:0:\cdots:0]$.

5 Various

The exercises in this section are independent.

Exercise For X an affine scheme, prove that $\neg\neg(x = y)$ if and only if:

$$\prod_{U : X \to \mathrm{Open}} U(x) \to U(y)$$

Same for X any scheme.