Differential Geometry of Synthetic Schemes

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Contents

Introduction

We work with a fixed ring R, which is commutative and local. We assume throughout, that synthetic quasi coherence (SQC) and Zariski-local choice (Z-choice) hold, like they are defined in [CCH23].

1 Tangent Spaces

1.1 The Tangent Space

We will use the concept of tangent spaces from synthetic differential geometry. More concretely, we follow [Mye22][Section 4] on the subject, which also uses homotopy type theory as a basis.

Definition 1.1.1 The *first order disk* of dimension n is the type

$$\mathbb{D}(n) :\equiv \{x : R^n \mid xx^T = 0 : R^{n\times n}\}.$$

More generally, for an R-module V, the first order disk of V is the type

$$\mathbb{D}(V) :\equiv \{f : V^\star \mid \forall v\, v'.f(v)f(v') = 0\},$$

where $V^\star$ is the R-linear dual of V.

Note that $\mathbb{D}(n)$ is equivalent to $\mathbb{D}(R^n)$.

Definition 1.1.2 For V an R-module, the associated square-zero extension of R is the R-module $R \oplus V$ with multiplication given by

$$(r, v)(r', v') := (rr', rv' + r'v).$$

Note that if V is a finitely presented R-module, then so is $R \oplus V$. Hence in particular $R \oplus V$ is a finitely presented R algebra in this case.

Lemma 1.1.3 For any R-module V, we have an equivalence

$$\mathrm{Spec}(R \oplus V) \simeq \mathbb{D}(V)$$

Proof An R-algebra morphism $R \oplus V \to R$ is determined by an R linear map $R \oplus V \to R$ that respects multiplication. Equivalently, this is an R-linear map $f : V \to R$ such that $rr' + f(rv' + r'v) = rr' + r'f(v) + rf(v') + f(v)f(v')$. By linearity, this condition amounts to $f(v)f(v') = 0$. □

Thus $\mathbb{D}(V)$ is an affine scheme whenever V is finitely presented. In particular $\mathbb{D}(n)$ is always an affine scheme. More generally than first order disks, we can consider infinitesimal varieties:

Definition 1.1.4 (a) A *Weil algebra* over R is a finitely presented R-algebra W together with a homomorphism $\pi : W \to R$, such that the kernel of π is a nilpotent ideal.

(b) An *infinitesimal variety* is a pointed type D, such that $D = (\mathrm{Spec}\, W, \pi)$ for a Weil algebra (W, π).

Note that the kernel of π is a finitely generated ideal, as the kernel of a surjective homomorphism between finitely presented algebras. To ask that $\ker \pi$ is nilpotent as an ideal is therefore the same as to ask that each of its elements is nilpotent.

Lemma 1.1.5 (using ??, ??) A pointed affine scheme $(\mathrm{Spec}\, A, \pi)$ is an infinitesimal variety if and only if, for every $x : \mathrm{Spec}\, A$ we have $\neg\neg(x = \pi)$.

Proof First note that we can choose generators $X_1, \dots, X_n$ of A such that $\pi(X_i) = 0$ for all i (by replacing X_i with $X_i - \pi(X_i)$ if necessary) and therefore $\ker \pi = (X_1, \dots, X_n)$.

Assume $(\operatorname{Spec} A, \pi)$ is an infinitesimal variety. By (**??**), this means that (A, π) itself is a Weil algebra, so every X_i is nilpotent in A. Now if $x : \operatorname{Spec} A$ is any homomorphism $A \to R$, then $x(X_i)$ is also nilpotent in R, meaning, by **??**, that $\neg\neg(x(X_i) = 0)$. Since we have this for all $i = 1, \dots, n$ and double negation commutes with finite conjunctions, we have $\neg\neg(x = \pi)$.

Now assume $\neg\neg(x = \pi)$ for all $x : \operatorname{Spec} A$. To show that (A, π) is a Weil algebra, let $f : A$ be given with $\pi(f) = 0$. Then in particular we have $\neg\neg(x(f) = 0)$ for every $x : \operatorname{Spec} A$. But this means $D(f) = 0$ (using (**??**) for $\operatorname{inv}(x(f)) \Rightarrow x(f) \neq 0$), so f is nilpotent by **??**. □

The following lemma allows us to reduce maps from infinitesimal varieties to schemes to the affine case:

Lemma 1.1.6 Let X be a scheme, V an infinitesimal variety and $p : X$. Then for all affine open $U \subseteq X$ containing p, there is an equivalence of pointed mapping types:

$$V \to_{\text{pt}} (X, p) \quad \cong \quad V \to_{\text{pt}} (U, p)$$

Proof By lemma 1.1.5, all points in V are not not equal, so all points in the image of a pointed map

$$V \to_{\text{pt}} (X, p)$$

will be not not equal to p. Since $p \in U$ and open propositions are double-negation stable (**??**), the image is contained in U and the map factors uniquely over (U, p). □

Definition 1.1.7 Let X be a type.

(a) For $p : X$ a point in X. The *tangent space* of X at p, is the type

$$T_p X :\equiv \{t : \mathbb{D}(1) \to X \mid t(0) = p\}.$$

(b) The *tangent bundle* of X is the type $X^{\mathbb{D}(1)}$.

Note that for any map $f : X \to Y$, we have a map $Df_p : T_p X \to T_{f(p)} Y$ given by $Df_p(t, x) = f(t(x))$. This makes tangent spaces functorial.

Definition 1.1.8 For A an R-algebra and M an A-module, a *derivation* is an R-linear map $d : A \to M$ satisfying the Leibniz rule

$$d(fg) = f \cdot dg + g \cdot df.$$

Lemma 1.1.9 For A an R-algebra and V an R-module, we have an equivalence between R-algebra maps $A \to R \oplus V$ and pairs (p, d) where $p : A \to R$ is an R-algebra map, and $d : A \to V$ is a derivation, where the A-module structure on V is obtained by restricting scalars along p.

Proof An R-algebra map is $A \to R \oplus V$ is given by an R-algebra map $p : A \to R$ and an R-linear map $d : A \to V$ such that $a \mapsto (p(a), d(a))$ respects multiplication. Since p respects multiplication and d is linear, this amounts to the Leibniz rule. □

We transfer a result of Myers [Mye22][Theorem 4.2.19] to schemes:

Theorem 1.1.10 (using ??)
Let X be a scheme and $p : X$ a point. Then $T_p X$ is an R-module.

We emphasize that this is an untruncated statement: we construct an R-module structure on $T_p X$, as opposed to showing the mere existence of an R-module structure.

Proof Following the proof of [Mye22][Theorem 4.2.19], it is enough to show that any scheme is infinitesimal linear in the sense that

$$\begin{array}{ccc} X^{\mathbb{D}(n+m)} & \longrightarrow & X^{\mathbb{D}(n)} \\ \big\downarrow & & \big\downarrow \\ X^{\mathbb{D}(m)} & \longrightarrow & X \end{array}$$

is a pullback for all $n, m : \mathbb{N}$. This amounts to showing that for any point $p : X$, the map

$$(\mathbb{D}(n+m) \to_{\mathrm{pt}} X) \to (\mathbb{D}(n) \to_{\mathrm{pt}} X) \times (\mathbb{D}(m) \to_{\mathrm{pt}} X)$$

is an equivalence. Since this statement is a proposition, we may assume given a an affine open $U = \operatorname{Spec} A$ containing p. By lemma 1.1.6, we may assume $X = U$. In this case, $\mathbb{D}(n) \to_{\mathrm{pt}} X$ is equivalent to the type of derivations $A \to R^n$, where the A-module structure on R^n is obtained by restricting scalars along $p : A \to R$. It is clear that a derivation $A \to R^n \oplus R^m$ is described by a pair of derivations $A \to R^n$, $A \to R^m$, as needed. □

We can characterise addition $+ : T_pX \to T_pX \to T_pX$ on tangent spaces as follows. Given $f, g : T_pX$, there is a unique $h : \mathbb{D}(2) \to X$ such that $h(x, 0) = f(x)$, $h(0, y) = g(y)$ for all $x, y : \mathbb{D}(1)$. Then $(f + g)(x) = h(x, x)$. The scalar action of R on T_pX is given by $(rf)(x) = f(rx)$. This makes it clear that $Df_p : T_pX \to T_pX$ is R-linear.

For an alternative characterisation, we note that for any map $q : X \to \mathbb{A}^1$, we have $q((f + g)(x)) = q(f(x)) + q(g(x))$. This determines $f + g : \mathbb{D}(1) \to X$ uniquely in the case where points of X are separated by functions $X \to \mathbb{A}^1$ (e.g. if X is an affine scheme).

Lemma 1.1.11 For any $p : \mathbb{A}^n$, the map $R^n \to T_p\mathbb{A}^n$ given by $v \mapsto x \mapsto p + xv$ is an isomorphism of R-modules.

Proof It is direct that the map is R-linear. It remains to show it is an equivalence. By considering each component separately, we may assume $n = 1$. In this case $T_p\mathbb{A}^1$ corresponds by SQC to elements of $R \oplus R$ whose first component is zero. We omit the verification that the map is the one we described. □

Definition 1.1.12 We say an R-module V is *finitely co-presented* if it can merely be represented as the kernel of some linear map $R^n \to R^m$ of finite free modules.

Equivalently, an R-module is finitely co-presented if it is the dual of some finitely presented R module.

Lemma 1.1.13 Let X be a scheme and $p : X$ a point. Then T_pX is finitely co-presented.

Explicitly, if X is affine and given by the following pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{A}^n & \longrightarrow & \mathbb{A}^m \end{array}$$

then T_pX is given by a pullback diagram of R-modules

$$\begin{array}{ccc} T_pX & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ R^n & \longrightarrow & R^m \end{array}$$

Proof If X is a general scheme, we reduce to the affine case by picking an affine patch. The affine case follows from the fact that pointed exponentiation with $\mathbb{D}(1)$ preserves pullback squares. □

Since finitely co-presented modules are affine schemes, it follows that any tangent space of a scheme is an affine scheme. Since schemes are closed under sigma-types, the tangent bundle of any scheme is again a scheme.

The following is a completely general algebraic fact about Taylor expansions of polynomials.

Lemma 1.1.14 For any map $f : \mathbb{A}^n \to \mathbb{A}^m$, the Jacobian of f is an $n \times m$ matrix describing a linear map $Jf_p : R^n \to R^m$, such that for all $p : \mathbb{A}^n$, $x : \mathbb{D}(1)$, $v : R^n$, we have

$$f(p + xv) = f(p) + xJf_pv.$$

It follows that the derivative $R^n \to R^m$ of a map $\mathbb{A}^n \to \mathbb{A}^m$ at a point of $\mathbb{A}^n$ is given by the Jacobian matrix. In this way we can effectively compute the tangent space of an affine scheme. For example, since the Jacobian of a linear map is that same linear map, we see that any tangent space of a finitely co-presented R-module is naturally that same R-module.

Lemma 1.1.15 Let M, N be finitely presented modules. Then linear maps $M \to N$ correspond to pointed maps $\mathbb{D}(N) \to_{\mathrm{pt}} \mathbb{D}(M)$. Explicitly, a linear map $g : M \to N$ corresponds to the pointed map $f \mapsto m \mapsto f(g(m))$.

Proof Pointed maps $\mathbb{D}(N) \to_{\mathrm{pt}} \mathbb{D}(M)$ correspond to R-algebra maps $R \oplus M \to R \oplus N$ lifting the projection $R \oplus M \to R$, and hence to derivations $R \oplus M \to N$, where the $R \oplus M$-module structure on M is obtained by restricting scalars along the projection $R \oplus M \to R$. The Leibniz rule amounts to $dr = 0$ for $r : R$, so we obtain all R-linear maps $M \to N$ in this way. □

Lemma 1.1.16 Every finitely presented R-module is naturally isomorphic to its double dual. Hence every finitely co-presented R-module is naturally isomorphic to its double dual. Hence taking R-linear duals is a self-inverse contravariant equivalence of categories between finitely presented and finitely co-presented R-modules.

Note that this is true for discrete fields but wildly false for general rings. For example $\mathbb{Z}/2\mathbb{Z}$ is a finitely presented $\mathbb{Z}$-module whose double dual is zero.

Proof Let M be a finitely presented R-module, and let $c : M^\star \to R$ be an element of the double dual. We can restrict c to a map $\mathbb{D}(M) \to_{\mathrm{pt}} R$. By SQC, this corresponds to an element of $R \oplus M$ whose first component is zero. It remains to check that this determines an equivalence $M^{\star\star} \simeq M$. By construction, c is sent to $m : M$ if and only if for all $f : \mathbb{D}(M)$, $c(f) = f(m)$. It remains to show that this condition is equivalent to the condition that for all $f : M^\star$, $c(f) = f(m)$. The reverse implication is clear. Thus let us consider the forward implication. Suppose $f : M^\star$ and we want to show $c(f) = f(m)$ in R. By SQC it suffices to show $xc(f) = xf(m)$ for all $x : \mathbb{D}(1)$. Indeed $xc(f) = c(xf) = xf(m)$ by linearity and using the fact that $xf : \mathbb{D}(M)$. □

Lemma 1.1.17 For V a finitely co-presented R-module, V is the spectrum of the free R-algebra on the R-module $V^\star$, i.e. of the symmetrisation of the tensor algebra on $V^\star$.

Proof By universal properties, the relevant spectrum is equivalent to the type of R-module maps $V^\star \to R$, i.e. to V. □

1.2 Cotangent spaces

Definition 1.2.1 For X a type and $p : X$ a point, the *cotangent space* at p is the R-linear dual $T_p^\star X$ of the tangent space $T_p X$.

If X is a scheme, then by lemma 1.1.16 the cotangent spaces of X are finitely presented.

Definition 1.2.2 For A an R-module, the type $\Omega_{A/R}$ of *Kähler differentials* is the codomain of the universal derivation $d : A \to \Omega_{A/R}$.

That is, $\Omega_{A/R}$ is generated as an A-module by symbols df, $f : A$, subject to relations $d(rf) = r \cdot df$ for $r : R$ and $d(fg) = f \cdot dg + g \cdot df$. It can be seen that if A is finitely presented as an R-algebra, then $\Omega_{A/R}$ is finitely presented as an A-module.

Lemma 1.2.3 For $X = \operatorname{Spec} A$ an affine scheme, recall from [CCH23, Theorem 8.2.3] that there is an equivalence of categories between finitely presented A-modules and families of finitely presented R-algebras over X. Under this correspondence, $\Omega_{A/R}$ corresponds to the cotangent bundle of X.

Proof We need to show that for $p : X$, we have an isomorphism of R-modules

$$\Omega_{A/R} \otimes_A R \simeq T_p^\star X.$$

Recall that the tangent space space $T_p X$ corresponds to derivations $A \to R$, where the A-module structure on R is obtained from p. These correspond to A-module maps $\Omega_{A/R} \to R$, by the universal property of Kähler differentials. By the restriction-extension of scalars adjunction, these correspond to R-linear maps $\Omega_{A/R} \otimes_A R \to R$. Thus $T_p X$ is the dual of $\Omega_{A/R} \otimes_A R$. By lemma 1.1.16 we get the desired result. □

2 Formal neighborhoods

Recall from [CCH23] that for a type X with $x : X$, we say that $y : X$ is in the *formal neighborhood* of x if $\neg\neg(x = y)$. We will write $N_\infty(x)$ for the type of $y : X$ such that $\neg\neg(x = y)$.

Definition 2.0.1 For a type X and $k : \mathbb{N}$, we say $x, y : X$ are *neighbors of order* k if $\neg\neg(x = y)$ and for all $U \subseteq X$ open containing x (and hence y), and $q_0, \ldots, q_k : U \to R$, we have $(q_0(x) - q_0(y)) \cdots (q_k(x) - q_k(y)) = 0$. This is equivalent to $q_0(y) \cdots q_k(y) = 0$ for all $q_i : U \to R$ such that $q_i(x) = 0$. We write $N_k(x)$ for the type of all order k neighbors of x.

Note that this defines a symmetric and reflexive relation on X. It is functorial in the sense that if x, y are neighbors of order k and $f : X \to Y$ is some map, then $f(x)$ and $f(y)$ are also neigbors of order k. The relation is not transitive, but we do have that if x, y are neighbors of order k and y, z are neighbors of order l, then x and z are neighbors of order $k + l$. If X is a scheme, then x, y are order 0 neighbors if and only if $x = y$.

Lemma 2.0.2 Let $X = \operatorname{Spec} A$ be an affine scheme and suppose $f_1, \ldots, f_n$ generate A. Then x and y are neighbors of order k iff $(f_1(x) - f_1(y))^{e_1} \cdots (f_n(x) - f_n(y))^{e_n} = 0$ for any $e_1, \ldots, e_n : \mathbb{N}$ with $e_1 + \ldots + e_n = k + 1$.

Proof The forward implication is direct, so let us consider the reverse. We may suppose $f_i(x) = 0$, since $f_i - f_i(x)$ also forms a generating set of A. In particular $f_i(y)$ is nilpotent, so not not zero. So not not every $f_i(y)$ is zero. This shows that not not $x = y$, since points of an affine scheme are separated by functions $X \to R$.

Now let $U \subseteq X$ be an open neighborhood of x and $q_0, \ldots, q_k : U \to R$ with $q_i(x) = 0$. We can write U as $D(g_1, \ldots, g_m)$ with $g_j : A$. Since $x \in U$ we have $x : D(g_j)$ for some j. Write $g = g_j$. Now q_i restricts to a map $D(g) \to R$, which corresponds to an element $p_i : A_g$. Write $p_i = a_i/g^l$ with $a_i : A$. We want to show $a_0(y) \cdots a_k(y) = 0$. By assumption we can write each a_i as a polynomial in $f_1, \ldots, f_n$ with zero constant term. Thus $a_0(y) \cdots a_k(y)$ is a sum of monomials of degree at least $k + 1$ in $q_1, \ldots, q_n$. Since each monomial vanishes, so does the sum. □

As a corollary, we have that $N_k(x)$ is an affine scheme for any point x of a scheme; indeed it is a closed subscheme of any open affine neighborhood of x. We give another criterion:

Lemma 2.0.3 Let $X = \operatorname{Spec} A$ be an affine scheme. Then x and y in X are neighbors of order k iff there merely exists a fg ideal I in R such that $I^{k+1} = 0$ and $I = 0 \to x =_X y$.

Proof Given such an ideal I and $q_0, \cdots, q_n : U \to R$ with $x, y \in U$, we have that $I = 0$ implies $q_j(x) = q_j(y)$, i.e. $q_j(x) - q_j(y) \in I$, so that:

$$(q_0(x) - q_0(y)) \cdots (q_k(x) - q_k(y)) \in I^{k+1} = 0$$

and x and y are neighbours of order k.

Conversely assume x and y are neighbours of order k. If $f_0, \cdots, f_n$ generate A, then $I = (f_0(x) - f_0(y), \cdots, f_n(x) - f_n(y))$ satisfies the condition by lemma 2.0.2. □

Lemma 2.0.4 Let X be a scheme and $x, y : X$. If y is in the formal neighborhood of x, then there merely exists $k : \mathbb{N}$ such that y is in $N_k(x)$.

Proof Without loss of generality $X = \operatorname{Spec} A$ is affine. Pick generators $f_1, \ldots, f_n$ of A. Then $\neg\neg(f_i(x) = f_i(y))$. So $f_i(x) - f_i(y)$ is nilpotent for each i. Say $(f_i(x) - f_i(y))^{k_i+1} = 0$. Then x and y are order $k_1 + \ldots + k_n$ neighbors, since if $e_1 + \ldots + e_n = k_1 + \ldots + k_n + 1$, we have $e_i \geq k_i + 1$ for some i. □

Definition 2.0.5 For X a type and $x : X$ a point, the stalk $\mathcal{O}_x$ is the (filtered) colimit of $U \to R$ over open neighbourhoods $x \in U \subseteq X$.

As a warning, note that we cannot expect $\mathcal{O}_x$ to be finitely presented. If $X = \operatorname{Spec} A$ is affine, then $\mathcal{O}_x$ is the localisation of A away from the kernel of $x : A \to R$. There is a natural map $\mathcal{O}_x \to R$ of R-algebras, evaluating a germ at x.

Lemma 2.0.6 The map $\mathcal{O}_x \to R$ reflects invertible elements. In particular $\mathcal{O}_x$ is a local ring.

Proof Consider $x \in U \subseteq X$, $f : U \to R$. Suppose $f(x)$ is invertible. Then $\{y : U \mid f(y) \neq 0\}$ is also an open neighborhood of x, and f is invertible on it. Hence f is invertible in $\mathcal{O}_x$. □

Definition 2.0.7 The kernel of the evaluation map $\mathcal{O}_x \to R$ is the 'maximal ideal' $\mathfrak{m}_x$.

Lemma 2.0.8 If X is a scheme with $x : X$, then $N_k(x)$ is the spectrum of $\mathcal{O}_x/\mathfrak{m}_x^{k+1}$. In particular the latter is finitely presented over R.

Proof The map from $N_k(x)$ neighboorhood can be described directly, by evaluating at y. To see this is an equivalence, we suppose without loss of generality that $X = \operatorname{Spec} A$ is affine. Then $\mathcal{O}_x/\mathfrak{m}_x^{k+1} = A/(\mathfrak{m}_x \cap A)^{k+1}$, essentially since if $f : A$ with $f(x) \neq 0$, then we have that f is invertible in $A/(\mathfrak{m}_x \cap A)^{k+1}$, by the formula $(1-g)(1+g+\ldots+g^k) = 1$ modulo g^{k+1}. The spectrum of the latter is clearly the order k neighborhood of x. □

Lemma 2.0.9 For V a finitely presented R-module, the disk $\mathbb{D}(V)$ is the first order neighborhood of 0 in $V^\star$.

Proof By lemma 1.1.17, the algebra $V^\star \to R$ is generated by elements $f \mapsto f(v)$ for $v : V$. The result follows from lemma 2.0.2. □

Lemma 2.0.10 Let X be a scheme and $x : X$ a point. Then $N_1(x)$ is equivalent to the disk $\mathbb{D}(T_x^\star X)$. Moreover, we have an isomorphism of R-modules

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \simeq T_p^\star X.$$

This means that $x : X$ and $0 : T_pX$ have the same first order neighborhoods, which aligns well with an intuitive understanding of tangent spaces.

Proof The ring of functions $N_1(x) \to R$ is equivalent to $\mathcal{O}_x/\mathfrak{m}_x^2$, which is equivalent to the square-zero extension $R \oplus \mathfrak{m}_x/\mathfrak{m}_x^2$. Any $r : R$ determines an endomorphism of $\mathfrak{m}_x/\mathfrak{m}_x^2$ (multiplication by r), and hence an endomorphism of $\mathcal{O}_x/\mathfrak{m}_x^2$, and hence an endomorphism of $N_1(x)$, which by abuse of notation we write $y \mapsto (1-r)x + ry$. The defining property is that for $f : N_1(x) \to R$, we have $f((1-r)x+ry) = (1-r)f(x) + rf(y)$. Given $y : N_1(x)$, we define a tangent vector $\mathbb{D}(1) \to_{\text{pt}} (X, x)$ by $t \mapsto (1-t)x + ty$. This defines a map $N_1(x) \to_{\text{pt}} (T_xX, 0)$. It lifts to a map $N_1(x) \to_{\text{pt}} \mathbb{D}(T_x^\star X)$ by functoriality of N_1.

Next, we define a map $(N_1(x) \to R) \to \mathbb{D}(T_x^\star X) \to R$. Suppose $f : N_1(x) \to R$ and $v : \mathbb{D}(T_x^\star X)$. Then v restricts to a map $\mathbb{D}(1) \to_{\text{pt}} N_1(x)$ (as does any tangent vector), so we have $f \circ v : \mathbb{D}(1) \to_{\text{pt}} (R, f(x))$. By SQC, we can write $f(v(t)) = f(x) + ct$ for a well-defined $c : R$. We claim the map $f \mapsto v \mapsto f(x) + c$ respects multiplication; it is clear that it is R-linear in f. Thus suppose $f, f' : N_1(x) \to R$. Note that $(ff')(v(t)) = (f(x)+ct)(f'(x)+c't) = (ff')(x) + (f(x)c' + f'(x)c)t$, since $t^2 = 0$. It remains to show that $(ff')(x) + f(x)c' + f'(x)c = (f(x)+c)(f'(x)+c')$, i.e. that $cc' = 0$. This follows from the fact that v is in the first order neighbourhood of 0.

This defines maps back and forth between $N_1(x)$ and $\mathbb{D}(T_x^\star X)$. We omit the verification that they are inverse to each other. The claim that $\mathfrak{m}_x/\mathfrak{m}_x^2 = T_p^\star X$ follows from the fact that a module can be recovered from its square-zero extension. □

The axiom of synethetic quasi-coherence applies only to finitely presented R-algebras, and would be false for general R-algebras. Surprisingly, it is still true in the following special case.

Definition 2.0.11 For X a type and $x : X$ a point, let $\widehat{\mathcal{O}}_x$ denote the completion of $\mathcal{O}_x$ at the ideal $\mathfrak{m}_x$. That is, $\widehat{\mathcal{O}}_x$ is the inverse limit of $R \leftarrow \mathcal{O}_x/\mathfrak{m}_x \leftarrow \mathcal{O}_x/\mathfrak{m}_x^2 \leftarrow \ldots$

For example, if $X = \mathbb{A}^n$, then $\widehat{\mathcal{O}}_x$ is the power series ring in n variables.

Lemma 2.0.12 For X a scheme and $p : X$ a point, $\widehat{\mathcal{O}}_p$ is the ring of functions on the formal neighborhood $N_\infty(p)$ of p. Conversely, $N_\infty(p)$ is the spectrum of $\widehat{\mathcal{O}}_p$.

Proof Without loss of generality we assume X is affine. Since $N_\infty(p)$ is the sequential colimit of $N_k(p)$ over $k : \mathbb{N}$ by lemma 2.0.4, the ring of functions $N_\infty(p) \to R$ is indeed the limit of the rings of functions $N_k(p) \to R$, which is $\mathcal{O}_p/\mathfrak{m}_p^{k+1}$ by lemma 2.0.8.

It remains to show that any R-algebra homomorphism $\widehat{\mathcal{O}}_p \to R$ is given by evaluation at some $y : N_k(p)$. That is, given $f : \widehat{\mathcal{O}}_p \to R$, we need to show that f factors through $\mathcal{O}_p/\mathfrak{m}_p^k$ for some k. Let $\mathfrak{m}_p$ be generated by $X_1, \ldots X_n$. We claim that $\neg\neg(f(X_i) = 0)$. Indeed suppose $f(X_i) \neq 0$, so that $f(X_i)$

is invertible. Say $yf(X_i) = 1$ with $y : R$. Now $Z := 1 + yX_i + y^2X_i^2 + \dots$ is a well-defined element of $\widehat{\mathcal{O}}_p$, with $Z = 1 + yX_iZ$. Hence $f(Z) = 1 + yf(X_i)f(Z) = 1 + f(Z)$. This means $1 = 0$ in $\widehat{\mathcal{O}}_p$, which means $1 = 0$ in R, which is impossible.

Thus $\neg\neg(f(X_i) = 0)$, so $f(X_i)$ is nilpotent in R. Say $f(X_i)^{k_i+1} = 0$. Then $f(\mathfrak{m}_x^{k+1}) = 0$ where $k = k_1 + \dots + k_n$. So f factors through $\mathcal{O}_x/\mathfrak{m}_x^{k+1}$, as needed. □

3 About lifting and modalities

3.1 Generalities on modalities

In this section we state useful facts about modalities in HoTT. We assume L a left exact modality and L' the modality corresponding to L-separated types, i.e. types with L-modal identity types. We have in mind L the formally étale modality and L' the formally unramified modality. Precise references should be added.

Lemma 3.1.1 If L is accessible (i.e. defined by unique litfting conditions), then L' is defined by the corresponding 'at most one' lifting conditions.

Lemma 3.1.2 The modality L preserves n-types for any n.

The following is one of the many equivalent characterisations of left exact modalities.

Lemma 3.1.3 For any X the localisation $\eta_X : X \to L(X)$ induces L-localisations:

$$x =_X y \to \eta_X(x) =_{LX} \eta_X(y)$$

Proposition 3.1.4 A map $f : X \to Y$ is an L'-localisation if and only if f is surjective and for all $x, y : X$ the map:

$$ap_f : x =_X y \to f(x) =_Y f(y)$$

is an L-localisation.

Corollary 3.1.5 The modality L' preserve n-types for any n.

Corollary 3.1.6 If X is L'-modal then:

$$X \to LX$$

is an embedding.

This means that the successive localisations:

$$X \to L'X \to LX$$

are the image factorisation of the localisation $X \to LX$.

3.2 Lifting Properties

Lemma 3.2.1 If a map f has the left lifting property with respect to $l : A_0 \to A_1$ and $l' : A_1 \to A_2$, then f has the left lifting property with respect to $l' \circ l$.

Lemma 3.2.2 Assume given a commutative square of the form:

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[f]{} & Y \end{array}$$

Then the following are equivalent:

(i) A lift of the square.

(ii) For all $b : B$ a lift of:

$$\begin{array}{ccc} \mathrm{fib}_u(b) & \xrightarrow{g} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[f(b)]{} & Y \end{array}$$

(iii) For all $b : B$ a lift of:

$$\begin{array}{ccc} \mathrm{fib}_u(b) & \xrightarrow{g} & \mathrm{fib}_p(f(b)) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array}$$

Proof We have that (ii) is equivalent to (iii) by definition of the fiber.

We can assume that the square is of the form:

$$\begin{array}{ccc} \sum_{b:B} P_b & \longrightarrow & \sum_{y:Y} Q_y \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & Y \end{array}$$

where the top map is:

$$\lambda(b, p).(f(b), g(b, p))$$

for some $g : \prod_{b:B} P_b \to Q_{f(b)}$. A lift of this square is the same as an inhabitant of:

$$\prod_{b:B} \sum_{q:Q_{f(b)}} \prod_{p:P_b} g(b, p) = q$$

which is equivalent to (iii). □

Lemma 3.2.3 Assume given maps $u : A \to B$ and $p : X \to Y$. Then if p has the right lifting property (resp. at most one lift) against $\mathrm{fib}_u(b) \to 1$ for all $b : B$, then it has the right lifting property (resp. at most one lift) against u.

Proof By lemma 3.2.2 a lift of a square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

is equivalent a family of lifts of squares of the form:

$$\begin{array}{ccc} \mathrm{fib}_u(b) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \longrightarrow & Y \end{array}$$

But a product of contractible types (resp. propositions) is itself contractible (resp. a proposition) so we can conclude. □

Lemma 3.2.4 A map p has the right lifting property (resp. at most one lift) against a map $P \to 1$ if and only all the fibers of p have this property against $P \to 1$.

Proof This is an immediate consequence of the universal property of the fibers. □

4 Étale morphisms

4.1 Definition

Example 4.1.1 Let us assume $2 \neq 0$ in R and look at the following square:

$$\begin{array}{ccc} 1 & \longrightarrow & \mathbb{A}^1 \setminus \{0\} \\ \big\downarrow & & \big\downarrow{\scriptstyle x \mapsto x^2} \\ \mathbb{D}(1) & \longrightarrow & \mathbb{A}^1 \setminus \{0\} \end{array}$$

As the bottom map, we choose the inclusion of $\{x : R^\times \mid (x-1)^2 = 0\}$. Then, for any choice of the top map, there is a unique lift in this square:

$$\begin{array}{ccc} 1 & \longrightarrow & \mathbb{A}^1 \setminus \{0\} \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle x \mapsto x^2} \\ \mathbb{D}(1) & \longrightarrow & \mathbb{A}^1 \setminus \{0\} \end{array}$$

A non-finitely generated version of the following definition is usually used to define *formally étale* maps of schemes [1] and then it is subsequently noted, that formally étale maps of finite presentation are étale. Since all of our schemes are of finite presentations, this should be a correct definition of étale morphism:

Definition 4.1.2 (a) A map $f : X \to Y$ is *formally étale*, if for all finitely presented R-algebras A and all finitely generated nilpotent ideals $N \subseteq A$, and all squares like below, there is a unique lift:

$$\begin{array}{ccc} \mathrm{Spec}(A/N) & \longrightarrow & X \\ {\scriptstyle \iota}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ \mathrm{Spec}\, A & \longrightarrow & Y \end{array}$$

– where $\iota : \mathrm{Spec}(A/N) \to \mathrm{Spec}\, A$ is induced by the quotient map.

(b) A map $f : X \to Y$ of schemes is *étale*, if it is formally étale.

Another way to phrase our definition of formally étale maps would be to say, that they are the maps with the right lifting property ([RSS20][definition 1.45]) with respect to "left" maps of the form $\mathrm{Spec}(A/N) \to \mathrm{Spec}\, A$. We will use some well known, general closure properties of left maps, starting with closure under composition:

Lemma 4.1.3 For A an algebra and N a nilpotent ideal in A, the map:

$$A \to A/N$$

can be factored as maps of the form:

$$B \to B/(b)$$

where $b : B$ is such that $b^2 = 0$.

Proof TODO □

Lemma 4.1.4 Having the right lifting property against the following class of maps is equivalent:

(i) Maps of the form $\mathrm{Spec}(A/N) \to \mathrm{Spec}(A)$ for A f.g. algebra and N a nilpotent ideal.

(ii) Maps of the form $P \to 1$ for P a closed dense proposition.

(iii) Closed dense embeddings of types.

(iv) Maps of the form $(\epsilon = 0) \to 1$ for $\epsilon : R$ such that $\epsilon^2 = 0$.

[1] In [EGAIV3][§17], the definition of formally étale maps ranges over arbitrary ideals, but uses the same lifting condition as below.

(v) Maps of the form $\mathrm{Spec}(A/(a)) \to \mathrm{Spec}(A)$ for A f.g. algebra and $a : A$ such that $a^2 = 0$.

Proof (i) implies (ii) because closed dense proposition are of the form $\mathrm{Spec}(R/N)$ for N a nilpotent ideal.

(ii) implies (iii) because of lemma 3.2.3.

(iii) implies (iv) because $\epsilon = 0$ is closed dense when ϵ is nilpotent.

(iv) implies (v) because of lemma 3.2.3, as the fiber of $\mathrm{Spec}(A/(a)) \to \mathrm{Spec}(A)$ over x is $a(x) = 0$.

(v) implies (i) because of lemma 4.1.3. □

Lemma 4.1.5 A map is formally étale if and only if all its fibers are formally étale.

Proof By lemma 3.2.4 and the characterisation (ii) from the previous lemma. □

From (ii) we even get that being formally étale is a lex modality, so we have the following:

Proposition 4.1.6 We have the following stability results:

- If X is any type and for all $x : X$ we have a formally étale type Y_x, then:

$$\prod_{x:X} Y_x$$

is formally étale.

- If X is formally étale and for all $x : X$ we have a formally étale type Y_x, then:

$$\sum_{x:X} Y_x$$

is formally étale.

- If X is formally étale then for all $x, y : X$ the type $x = y$ is formally étale.
- The type of formally étale types is formally étale.

Lemma 4.1.7 Let A be a finitely presented R-algebra and $N \subseteq A$ be finitely generated nilpotent. Then for $V :\equiv \mathrm{Spec}(A/N) \subseteq \mathrm{Spec}\, A$ the following holds:

(a) For all $x : \mathrm{Spec}\, A$, $\neg\neg V(x)$.

(b) If $V = \emptyset$, then $\mathrm{Spec}\, A = \emptyset$.

Proof (a) Let $x : \mathrm{Spec}\, A$ be given. The generators $n_1, \ldots, n_l$ of N are nilpotent functions, so in particular the elements $n_1(x), \ldots, n_l(x)$ of R are not not zero. This means precisely $\neg\neg V(x)$.

(b) Assume $V = \emptyset$ and $x : \mathrm{Spec}\, A$. We want to show the $\neg\neg$-stable proposition $\emptyset$, so we can assume $V(x)$, which is a contradiction. □

4.2 Examples

Proposition 4.2.1 Let P be a $\neg\neg$-stable proposition, then $P \to 1$ is formally étale.

Proof Direct application of lemma 4.1.7. □

Proposition 4.2.2 The map $\mathrm{Bool} \to 1$ is étale.

Proof We have to extend maps $f : \mathrm{Spec}(A/(a)) \to \mathrm{Bool}$, with $a^2 = 0$. Since $\mathrm{Bool} \subseteq R$, the map f yields an element $f : A/(a)$ and we have a lift $\tilde{f} : A$ with $f = \tilde{f} + ab$. By lemma 4.1.7, we have for any $x : \mathrm{Spec}\, A$, that $\neg\neg(\tilde{f}(x) = 0)$ or $\neg\neg(\tilde{f}(x) = 1)$.

By Z-choice or computation, we find a $n : \mathbb{N}$, such that $\tilde{f}^n(x) = 0$ or $\neg\neg(\tilde{f}^n(x) = 1)$. With the map $1 - \_ : R \to R$, we can achieve the same for 1. □

Proposition 4.2.3 Formally étale types are stable by sums, so that finite types are formally étale.

Proof Binary sums are dependent sums over the booleans, which are formally étale by proposition 4.2.2. □

Proposition 4.2.4 The type $\mathbb{N}$ is formally étale.

Proof Assume given a map:

$$P \to \mathbb{N}$$

for P a closed dense proposition. We want to show it factors uniquely through 1. It is clear there is at most one lift as $\mathbb{N}$ has decidable equality.

By boundedness the map merely factors through a finite type, which is formally étale by proposition 4.2.3 so it merely has a lift. □

Proposition 4.2.5 Let g be a polynomial in $R[X]$ such that for all $x : R$ we have that $g(x) = 0$ implies $g'(x) \neq 0$. Then:

$$\mathrm{Spec}(R[X]/g)$$

is formally étale.

Proof Assume given $\epsilon : R$ such that $\epsilon^2 = 0$, we try to prove there is a unique dotted lift to any:

$$\begin{array}{ccc} R/\epsilon & \longleftarrow & R[X]/g \\ \uparrow & \swarrow & \\ R & & \end{array}$$

We proceed in two steps:

- We prove there merely is a lift. We can assume $x : R$ such that $g(x) = 0$ module ϵ, say $g(x) = b\epsilon$. For all $y : R$ we have:

  $$g(x + y\epsilon) = g(x) + yg'(x)\epsilon$$

  Since $\not(g(x) = 0)$, we have that $g'(x) \neq 0$ so it is invertible. Then:

  $$y = -\frac{b}{g'(x)}$$

  gives a lift.

- We prove there is at most one lift. Assume $x, y : R$ two lifts, then $g(x) = g(y) = 0$ and $x = y$ modulo ϵ. Then:

  $$0 = g(x) = g(y + (x - y)) = g(y) + g'(y)(x - y) = g'(y)(x - y)$$

  Since $g'(y)$ is invertible, we have that $x = y$. □

We can generalise the previous example:

Proposition 4.2.6 Assume given $P_1, \cdots, P_n : R[X_1, \cdots, X_n]$, inducing:

$$P = (P_1, \cdots, P_n) : R^n \to R^n$$

Assume that for all $X : R^n$ such that $P(X) = 0$ we have that the jacobian matrix $J(X)$ is invertible. Then:

$$\mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_n)$$

is formally étale.

Proof Same as the previous lemma using the fact that for all $\epsilon : R$ such that $\epsilon^2 = 0$ and for all $X : R^n$ and $Y : (\epsilon)^n$ we have:

$$P(X + Y) = P(X) + J(X)Y$$

□

4.3 Separated étale schemes have decidable equality

Kind of outdated.

Proposition 4.3.1 Let X be an affine scheme and $a : X$ a point. Suppose the tangent space $T_a X$ has a unique point. Then for any $b : X$, equality $a = b$ is decidable.

Proof Given that $T_aX = \{0\}$, we also have $T_a^\star X = \mathfrak{m}_a/\mathfrak{m}_a^2 = 0$ by lemma 2.0.10. So $\mathfrak{m}_a^2 = \mathfrak{m}_a$. By [LQ15, Lemma II.4.6] (proved using Nakayama's lemms, or the determinant trick), $\mathfrak{m}_a$ is generated by a single idempotent e of A. We have $e(b)$ idempotent in R, so since R is local it is either 0 or 1. We have $e(a) = 0$, so if $e(b) = 1$ we have $a \neq b$. If $e(b) = 0$, then $a = b$ since b is in the order 0 neighborhood of a. □

Proposition 4.3.2 Let X be a separated étale scheme. Then X has decidable equality.

Proof Let $a, b : X$. Since X is separated, $a = b$ is closed. We claim it is also open. To see this, pick an affine open neighbourhood U of a. Now $a = b$ is equivalent to $(b \in U) \wedge (a = b)$. Since open propositions are closed under Σ, it suffices to show that $a = b$ is open assuming $b \in U$. In this case we can apply proposition 4.3.1. □

4.4 Hensel lifting

Lemma 4.4.1 Let $f : \mathbb{A}^1 \to \mathbb{A}^1$ and $p : \mathbb{A}^1$ with $f(p)$ nilpotent and $f'(p)$ invertible. Then there exists $q : \mathbb{A}^1$ with $\neg\neg(q = p)$ and $f(q) = 0$.

Proof Let $f(p)^n = 0$, induct on n. If $n \leq 1$ we are done. Otherwise, let $p' = p - f(p)/f'(p)$. Then $\neg\neg(p' = p)$ and we have

$$f(p') = f(p) - f'(p)(f(p)/f'(p)) + r(f(p)/f'(p))^2 = r(f(p)/f'(p))^2$$

for some $r : R$ by Taylor expansion. Thus $f(p')^m = 0$ for $2m \geq n$ and we are done by inductive hypothesis. □

4.5 Étale schemes are locally standard étale

Definition 4.5.1 An algebra is called standard étale if it is merely of the form:

$$(R[X_1, \cdots, X_n]/P_1, \cdots, P_n)_G$$

where $\det(\mathrm{Jac}(P_1, \cdots, P_n))$ divides G in $R[X_1, \cdots, X_n]/P_1, \cdots, P_n$.

Definition 4.5.2 A scheme is called standard étale if it is merely of the form $\mathrm{Spec}(A)$ for A a standard étale algebra.

Lemma 4.5.3 Standard étale schemes are étale.

Proof Assume given a standard étale algebra:

$$(R[X_1, \cdots, X_n]/P_1, \cdots, P_n)_G$$

and write:

$$P : R^n \to R^m$$

for the map induced by $P_1, \cdots, P_m$.

Assume given $\epsilon : R$ such that $\epsilon^2 = 0$, we need to prove that there is a unique dotted lifting in:

$$\begin{array}{ccc} R/\epsilon & \xleftarrow{x} & (R[X_1, \cdots, X_n]/P_1, \cdots, P_n)_G \\ \uparrow & \swarrow & \\ R & & \end{array}$$

This means that for all $x : R^n$ such that $P(x) = 0 \bmod \epsilon$ and $G(x)$ invertible modulo ϵ (or equivalently $G(x)$ invertible), there exists a unique $y : R^n$ such that:

- We have $x = y \bmod \epsilon$.
- We have $P(y) = 0$.
- We have $G(y) \neq 0$ (this is implied by $x = y \bmod \epsilon$ and $G(x) \neq 0$).

First we prove existence. For any $b : R^n$ we compute:

$$P(x + \epsilon b) = P(x) + \epsilon \; dP_x(b)$$

We have that $P(x) = 0 \bmod \epsilon$, say $P(x) = \epsilon a$. Then since $G(x) \neq 0$ and $\det(dP)$ divides G, we have that dP_x is invertible. Then taking $b = -(dP_x)^{-1}(a)$ gives a lift $y = x + \epsilon b$ such that $P(y) = 0$.

Now we check unicity. Assume y, y' two such lifts, then $y = y' \bmod \epsilon$ and we have:

$$P(y) = P(y') + dP_{y'}(y - y')$$

and $P(y) = 0$ and $P(y') = 0$ so that:

$$dP_{y'}(y - y') = 0$$

But $G(y') \neq 0$ so $dP_{y'}$ is invertible and we can conclude that $y = y'$. □

Proposition 4.5.4 Any étale scheme has a finite open cover by standard étale schemes.

Proof It is enough to prove this when the scheme is affine, say of the form:

$$\mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_m)$$

Consider the map:

$$P : R^n \to R^m$$

induced by $P_1, \cdots, P_m$. Since the fiber of P over 0 is étale, its tangent spaces are 0, meaning that the map:

$$\mathrm{Jac}(P_1, \cdots, P_m)_x = dP_x : R^n \to R^m$$

is injective for all:

$$x : \mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_m)$$

This means that one of the $n \times n$ minors of the jacobian is invertible, giving an open cover of:

$$\mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_m)$$

Up to rearranging the the order of the polynomials, we can assume that each piece merely is of the form:

$$\mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_n, Q_1, \cdots, Q_k)$$

where $\mathrm{Jac}(P_1, \cdots, P_n)_x$ is invertible for all x in the scheme.

Then we consider $x : \mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_n)$, we want to show that the proposition:

$$Q_1(x) = 0 \wedge \cdots \wedge Q_k(x) = 0$$

is decidable. To do this it is enough to prove that:

$$(Q_1(x), \cdots, Q_k(x))^2 = 0$$

implies:

$$(Q_1(x), \cdots, Q_k(x)) = 0$$

because then:

$$(Q_1(x), \cdots, Q_k(x))^2 = (Q_1(x), \cdots, Q_k(x))$$

and we conclude using Nakayama.

So let us assume:

$$(Q_1(x), \cdots, Q_k(x))^2 = 0$$

Then we consider:

$$\begin{array}{ccc} R/Q_1(x), \cdots, Q_k(x) & \xleftarrow{\;x\;} & R[X_1, \cdots, X_n]/P_1, \cdots, P_n, Q_1, \cdots, Q_k \\ \uparrow & \swarrow_{y} & \\ R & & \end{array}$$

We have a dotted lift because the scheme is étale, meaning we get $y : R^n$ such that:

$$y = x \bmod Q_1(x), \cdots, Q_k(x)$$

and for all i, j we have:

$$P_i(y) = 0,\ Q_j(y) = 0$$

Let us prove that $x = y$. Indeed $(x - y)(x - y)^t = 0$ so that:

$$P(x) = P(y) + (dP)_y(x - y)$$

but $P(x) = 0$, $P(y) = 0$ and $(dP)_y$ is invertible, so that $x = y$. From this we conclude that:

$$(Q_1(x), \cdots, Q_k(x)) = 0$$

as desired.

So at this point we have decomposed our scheme into decidable subtypes U of schemes of the form:

$$\mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_n)$$

where for all $x \in U$ we have:

$$\det(\mathrm{Jac}(P_1, \cdots, P_n)_x) \neq 0$$

Since U is decidable, it is in particular open so it is of the form $D(G_1, \cdots, G_n)$, and we have an open cover of our scheme by pieces of the form:

$$\mathrm{Spec}((R[X_1, \cdots, X_n]/P_1, \cdots, P_n)_G)$$

Where $P_i(x) = 0$ for all i and $G(x) \neq 0$ implies:

$$\det(\mathrm{Jac}(P_1, \cdots, P_n)_x) \neq 0$$

We write:

$$F(x) = \det(\mathrm{Jac}(P_1, \cdots, P_n)_x)$$

Then for all $x : \mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_n)$ we have that:

$$(G(x) \neq 0) \to (F(x) \neq 0)$$

so that there exists n such that:

$$F(x) | G(x)^n$$

and using boundedness we get N such that for all $x : \mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_n)$ we have:

$$F(x) | G(x)^N \qquad \square$$

and we conclude that F divides G^N in $R[X_1, \cdots, X_n]/P_1, \cdots, P_n$. So by replacing G by G^N, we get standard étale pieces.

5 Unramified morphisms

5.1 Definition

Definition 5.1.1 For any types X, Y, a map $f : X \to Y$ is called formally unramified if for any closed proposition P such that $\neg\neg P$ the following square has at most one lift:

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ 1 & \longrightarrow & Y \end{array}$$

Definition 5.1.2 A map $f : X \to Y$ is unramified if X and Y are schemes and f is formally unramified.

It is clear that formally étale maps are formally unramified. As usual a type X is called formally unramified if the map from X to 1 is unramified, and a map is formally unramified if and only if all its fibers are formally unramified.

Lemma 5.1.3 Having at most one lift against the following class of maps is equivalent:

(i) Maps of the form $\mathrm{Spec}(A/N) \to \mathrm{Spec}(A)$ for A f.g. algebra and N a nilpotent ideal.

(ii) Maps of the form $P \to 1$ for P a closed dense proposition.

(iii) Closed dense embeddings of types.

(iv) Maps of the form $(\epsilon = 0) \to 1$ for $\epsilon : R$ such that $\epsilon^2 = 0$.

(v) Maps of the form $\mathrm{Spec}(A/(a)) \to \mathrm{Spec}(A)$ for A f.g. algebra and $a : A$ such that $a^2 = 0$.

Proof Same as lemma 4.1.4. ☐

Lemma 5.1.4 A map is formally unramified if and only if all its fibers are formally unramified.

Proof By lemma 3.2.4 and the characterisation (ii) from the previous lemma. ☐

Lemma 5.1.5 A type X is formally unramified if and only if for any $x, y : X$ the type $x = y$ is formally étale.

Proof A type X is formally unramified iff for any closed dense proposition P the fibers of the canonical map in:

$$X \to X^P$$

are propositions. This is equivalent to the induced maps in:

$$(x = y) \to (x = y)^P$$

being equivalences for all $x, y : X$, i.e. all types $x = y$ being formally étale. ☐

In the langage of modalities, this means that formally unramified types are precisely formally étale-separated types. So being formally unramified is a (non-lex) modality, so that:

Proposition 5.1.6 We have the following stability results:

- If X is any type and for all $x : X$ we have a formally unramified type Y_x, then:

$$\prod_{x:X} Y_x$$

is formally unramified.

- If X is formally unramified and for all $x : X$ we have a formally unramified type Y_x, then:

$$\sum_{x:X} Y_x$$

is formally unramified.

5.2 Examples

Formally étale types are formally unramified.

Proposition 5.2.1 Any proposition is formally unramified.

This means any embedding is formally unramified.

Proposition 5.2.2 Subtype of formally étale types are formally unramified.

Proof By the previous example and stability by dependent sum. ☐

We will see all examples are of this form.

5.3 Unramified schemes

Proposition 5.3.1 A scheme X is unramified if and only if any of the following propositions hold:

(i) For all $x, y : X$, the proposition $x = y$ is open.

(ii) For all $x, y : X$, the proposition $x = y$ is $\neg\neg$-stable.

(iii) For all $x : X$, we have $T_x(X) = 0$.

(iv) For all infinitesimal pointed type $(D, *)$ (meaning that for all $x : D$ we have $\neg\neg(x = *)$), any map from D to X is constant.

Proof First we prove that the four propositions are equivalent:

(i) implies (ii) because open propositions are $\neg\neg$-stable.

(ii) implies (iv) because for any $f : D \to X$ and $x : D$ we have $\neg\neg(x = *)$ so that $\neg\neg(f(x) = f(*))$ and finally $f(x) = f(*)$.

(iv) implies (iii) by taking $D = \mathbb{D}(1)$.

(iii) implies (i) because for any $x : X$ there an open affine U such that $x \in U$. Then $x = y$ is equivalent to $(y \in U) \wedge x =_U y$, but $x =_U y$ is decidable by proposition 4.3.1 and open propositions are stable by Σ.

Now we check they are equivalent to being unramified:

(ii) implies unramified, indeed we need to check that for $x, y : X$ and P closed dense such that $P \to x = y$, we have $x = y$. But $\neg\neg P$ so that $\neg\neg(x = y)$, and by (ii) we have $x = y$.

Unramified implies (iii) because it implies having at most one lifting against any closed dense subtype, so that by considering $1 \subset \mathbb{D}(1)$ it implies having at most one tangent vector. □

5.4 Separated formally unramified schemes have decidable equality

Proposition 5.4.1 Let P be a closed, $\neg\neg$-stable proposition. Then P is decidable.

Proof Let P be the proposition $I = 0$ where I is a finitely generated ideal of R. We claim $(I^2 = 0) \to (I = 0)$. Indeed, if $I^2 = 0$, then no element of I can be invertible, so I is not not zero, and since P is $\neg\neg$-stable, $I = 0$. Hence I is generated by a single idempotent e of R. Since R is local, e is either 0 or 1. Since P is equivalent to $e = 0$, P is decidable. □

Proposition 5.4.2 Any separated unramified scheme has decidable equality.

Proof By proposition 5.3.1 its identity types are $\neg\neg$-stable. We conclude by proposition 5.4.1. □

5.5 Unramified maps between schemes

Lemma 5.5.1 For any map $f : X \to Y$ and $x : X$, we have that:

$$\mathrm{Ker}(df_x) = T_{(x, \mathrm{refl}_{f(x)})}(\mathrm{fib}_f(f(x)))$$

Proof This holds because:

$$(\mathrm{fib}_f(f(x)), (x, \mathrm{refl}_{f(x)}))$$

is the pullback of:

$$(X, x) \to (Y, f(y)) \leftarrow (1, *)$$

in pointed types, applied using $(\mathbb{D}(1), 0)$. □

Proposition 5.5.2 A map between schemes is unramified if and only if its differentials are injective.

Proof The map df_x is injective if and only if its kernel is 0. By lemma 5.5.1, this means that df_x is injective for all $x : X$ if and only if:

$$\prod_{x:X} T_{(x, \mathrm{refl}_{f(x)})}(\mathrm{fib}_f(f(x))) = 0$$

On the other hand having fibers with trivial tangent space is equivalent to:

$$\prod_{y:Y} \prod_{x:X} \prod_{p:f(x)=y} T_{(x,p)}(\mathrm{fib}_f(y)) = 0$$

Both are equivalent by path elimination on p. □

6 Smooth morphisms

6.1 Smooth maps

Definition 6.1.1 A morphism $f : X \to Y$ is formally smooth if for any closed dense proposition P, any square:

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ 1 & \longrightarrow & Y \end{array}$$

merely has a lift.

Lemma 6.1.2 For any morphism $f : X \to Y$ the following are equivalent:

(i) The map f is formally smooth.

(ii) For any $\epsilon : R$ such that $\epsilon^2 = 0$, there merely exists a lift to any square:

$$\begin{array}{ccc} \epsilon = 0 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ 1 & \longrightarrow & Y \end{array}$$

Proof It is clear that (i) implies (ii). Conversely any map $P \to 1$ for P dense closed can be decomposed as:

$$P_n \to P_{n-1} \to P_1 \to 1$$

where:

- For all k we have that P_k is the spectrum of a local ring so it has choice.
- For all k the map:
  $$P_k \to P_{k-1}$$
  is of the form:
  $$\mathrm{Spec}(A/a) \to \mathrm{Spec}(A)$$
  where $a^2 = 0$.

Then by (ii) we can merely find a lift pointwise to $P_k \to P_{k-1}$, and since P_{k-1} has choice we merely get a global lift. By iterating we merely get a lift to $P \to 1$. □

Remark 6.1.3 The usual definition for formally smooth is to ask for lifting in:

$$\begin{array}{ccc} \mathrm{Spec}(A/N) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

with N nilpotent. Note that here since we do not have:

$$\prod_{x:A} ||B(x)|| \to ||\prod_{x:A} B(x)||$$

we do not have a direct analogue to lemma 4.1.4 or lemma 5.1.3.

Our definition of formal smoothness is convenient because it implies:

Lemma 6.1.4 A map is formally smooth if and only if its fibers are formally smooth.

Proof The type of filler for:

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ 1 & \xrightarrow{y} & Y \end{array}$$

is equivalent to the type of filler for:

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{fib}_f(y) \\ \big\downarrow & \nearrow & \\ 1 & & \end{array}$$

□

Remark 6.1.5 Formally smooth and formally unramified implies formally étale.

6.2 Examples

We give a few examples and counter-examples:

Lemma 6.2.1 The scheme $\mathbb{A}^n$ is smooth for any n.

Proof We need to prove that there merely exists a dotted lift in any:

$$\begin{array}{ccc} R/N & \longleftarrow & R[X_1, \cdots, X_n] \\ \big\uparrow & \swarrow & \\ R & & \end{array}$$

It is enough to choose a lift for each X_i. □

Lemma 6.2.2 A type covered by finitely many formally smooth subtypes is formally smooth.

Proof We assume $P_1, \cdots, P_n : X \to \mathrm{Prop}$ covering X, i.e. for all $x : X$ we have:

$$\prod_{x:X} P_1(x) \vee \cdots \vee P_n(x)$$

such that $\Sigma_{x:X} P_i(x)$ is formally smooth for all i.

Assume given:

$$\begin{array}{ccc} P & \xrightarrow{\phi} & X \\ \big\downarrow & \nearrow & \\ 1 & & \end{array}$$

then we have:

$$\prod_{p:P} P_1(\phi(p)) \vee \cdots \vee P_n(\phi(p))$$

so that for some i we have:

$$\prod_{p:P} P_i(\phi(p))$$

as P is closed. So ϕ factors through $\Sigma_{x:X} P_i(x)$ which is formally smooth and we can conclude. □

Corollary 6.2.3 The scheme $\mathbb{P}^n$ is smooth for any n.

Lemma 6.2.4 The scheme $\mathbb{D}(1)$ is not smooth.

Proof If it were smooth, for any ϵ with $\epsilon^3 = 0$ we would be able to prove $\epsilon^2 = 0$. Indeed we would merely have a dotted lift in:

$$\begin{array}{ccc} R/(\epsilon^2) & \xleftarrow[\epsilon]{} & R[X]/(X^2) \\ \big\uparrow & \swarrow & \\ R & & \end{array}$$

that is, an $r : R$ such that $(\epsilon + r\epsilon^2)^2 = 0$. Then $\epsilon^2 = 0$. □

Lemma 6.2.5 The scheme $\mathrm{Spec}(R[X,Y]/(XY))$ is not smooth.

Proof If it were smooth, for any ϵ with $\epsilon^3 = 0$ we would be able to prove $\epsilon^2 = 0$. Indeed we would merely have a dotted lift in:

$$\begin{array}{ccc} R/(\epsilon^2) & \longleftarrow & R[X,Y]/(XY) \\ \uparrow & \swarrow & \\ R & & \end{array}$$

where the top map sends both X and Y to ϵ. Then we have $r, r' : R$ such that $(\epsilon + r\epsilon^2)(\epsilon + r'\epsilon^2) = 0$ so that $\epsilon^2 = 0$. ☐

Lemma 6.2.6 The map:

$$p : \text{Spec}(R[X,Y]/(XY)) \to \mathbb{A}^1$$

corresponding to the map:

$$R[X,Y]/(XY) \leftarrow R[X]$$

sending X to X is not smooth.

Proof If it were smooth all its fibers would be smooth, i.e. for all $z : R$ the scheme $\text{Spec}(R[X]/(zX))$ would be smooth. This would imply that for any $\epsilon : R$ such that $\epsilon^2 = 0$ we merely have a dotted lift in:

$$\begin{array}{ccc} R/\epsilon & \longleftarrow & R[X]/(\epsilon X) \\ \uparrow & \swarrow & \\ R & & \end{array}$$

where the top map sends X to 1. Such a lift gives an $r : R$ such that $\epsilon(1 + r\epsilon) = 0$, so that $\epsilon = 0$. ☐

I think in the traditional setting this map has smooth fibers, but not here.

6.3 Stability properties

Now we give stability properties for formally smooth types. We start by expected ones:

Lemma 6.3.1 If X is a type satifying choice and for all $x : X$ we have a formally smooth type Y_x, then:

$$\prod_{x:X} Y_x$$

is formally smooth.

So for example formally smooth types are stable by finite products.

Lemma 6.3.2 If X is a formally smooth type and for all $x : X$ we have a formally smooth type Y_x, then:

$$\sum_{x:X} Y_x$$

is formally smooth.

Formally smooth types are not stable by identity types (e.g. identity types in $\mathbb{A}^1$ are not smooth, otherwise they would be closed and étale, i.e. decidable).

One typically expects quotienting to sometimes break smoothness. Surprisingly, this is not the case in our setting when using homotopy quotients:

Proposition 6.3.3 The image of a formally smooth type by any map is formally smooth.

Proof We assume X formally smooth and $p : X \to Y$ surjective. Then for any P closed dense and any diagram:

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & \searrow & \uparrow p \\ 1 & \xrightarrow{x} & X \end{array}$$

by choice for closed propositions we merely get the dotted diagonal, and since X is formally smooth we get the dotted x, and then $p(x)$ gives a lift. $\square$

Lemma 6.3.4 Any pointed connected type X is formally smooth.

Proof We have a surjection:

$$1 \to X$$

and 1 is formally smooth so we conclude by proposition 6.3.3. $\square$

Formal smoothness is really about sets in some way:

Lemma 6.3.5 A type X is formally smooth if and only if its set truncation $\|X\|_0$ is formally smooth.

Proof Consider the map:

$$p : X \to \|X\|_0$$

- If X is formally smooth, then since p is surjective we conclude using proposition 6.3.3.
- The map p has merely inhabited connected fibers so by lemma 6.3.4 it is formally smooth, so that if $\|X\|_0$ is formally smooth so is X by lemma 6.3.2. $\square$

6.4 Equivalence with the usual definition for maps between schemes

Here we prove that our definition coincides with the usual one for maps with scheme fibers.

Lemma 6.4.1 Let X be a scheme and $\epsilon : R$ such that $\epsilon^2 = 0$. Then the type of liftings of:

$$\begin{array}{ccc} \epsilon = 0 & \xrightarrow{\phi} & X \\ \downarrow & & \\ 1 & & \end{array}$$

is an M-pseudotorsor where:

$$M = \mathrm{Hom}_{R/\epsilon}\Big(\prod_{p:\epsilon=0} T^\star_{\phi(p)}(X), (\epsilon)\Big)$$

Proof TODO $\square$

Lemma 6.4.2 The R-module M from the previous lemma is wqc.

Proof For any $p : \epsilon = 0$ we have that $T^\star_{\phi(p)}(X)$ is a finitely presented R-module, so that:

$$N = \prod_{p:\epsilon=0} T^\star_{\phi(p)}(X)$$

is a finitely presented R/ϵ-module. Assume a presentation:

$$(R/\epsilon)^m \to (R/\epsilon)^n \to N \to 0$$

then we have an exact sequence of R-modules:

$$0 \to \mathrm{Hom}_{R/\epsilon}(N, (\epsilon)) \to (\epsilon)^n \to (\epsilon)^m$$

but (ϵ) is wqc so that $\mathrm{Hom}_{R/\epsilon}(N, (\epsilon))$ is the kernel of a map between wqc R-modules and it is wqc. $\square$

Proposition 6.4.3 Let $p : X \to Y$ be a map which fibers are schemes. Then p merely having lifts against the following classes of maps is equivalent:

(i) The maps $\epsilon = 0 \to 1$ where $\epsilon^2 = 0$.

(ii) The maps $P \to 1$ where P is closed dense (i.e. p being formally smooth).

(iii) The maps $\mathrm{Spec}(A/a) \to \mathrm{Spec}(A)$ where A fp R-algebra and $a^2 = 0$.

(iv) The maps $\mathrm{Spec}(A/N) \to \mathrm{Spec}(A)$ where A fp R-algebra and N fg nilpotent ideal.

Proof It is enough to prove that (i) implies (iii), as any map in (iv) is a composite of maps in (iii). Assume a diagram:

$$\begin{array}{ccc} \mathrm{Spec}(A/a) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

with $a^2 = 0$, we try to merely find a lift. By lemma 6.4.1, we know that the type of lifts over $x : \mathrm{Spec}(A)$ is an M_x-pseudotorsor. By hypothesis (i) this is in fact an M_x-torsor. Mere existence of a lift for the diagram is then precisely a mere section of the dependent torsor $(x : \mathrm{Spec}(A)) \mapsto M_x$, i.e. a proof that it is merely trivial. But M_x is wqc by lemma 6.4.2 so that $H^1(\mathrm{Spec}(A), M) = 0$ by [CCH23][Theorem 8.3.6] and any M-torsor is merely trivial, meaning we merely have a lift. □

6.5 Smooth schemes and Jacobians (obsolete?)

Lemma 6.5.1 Assume given a smooth affine scheme:

$$X = \mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_m)$$

such that for all $x : X$ the Jacobian:

$$J(x) : R^n \to R^m$$

is surjective. Then X is smooth.

Proof We write $P : R^n \to R^m$ the map sending x to $(P_1(x), \cdots, P_m(x))$. Assume given $\epsilon : R$ such that $\epsilon^2 = 0$. For any:

$$\begin{array}{ccc} R/\epsilon & \longleftarrow & R[X_1, \cdots, X_n]/P_1, \cdots, P_m \\ \uparrow & \swarrow & \\ R & & \end{array}$$

we need to find a dotted lift. This means that given $x : R^n$ such that $P(x) = 0 \bmod \epsilon$, we need to merely find $y : R^n$ such that $P(x + \epsilon y) = 0$. But since $\epsilon^2 = 0$, we have that:

$$P(x + \epsilon y) = P(x) + \epsilon J(x) y$$

But we know that $P(x)$ is merely equal to ϵz for some z in R^m, so to conclude it is enough to prove $J(x)$ surjective. But $J(x)$ being surjective is an open proposition as it can be expressed as the invertibility of some determinants, so in particular it is $\neg\neg$-stable. Since $P(x) = 0$ implies $J(x)$ surjective and we have $\neg\neg(P(x) = 0)$ we can conclude. □

We want to show some kind of converse. We start with a simple case, when $m = 1$. First an auxiliary lemma.

Lemma 6.5.2 Assume given:

$$P : R[X_1, \cdots, X_n]$$

such that:

$$\mathrm{Spec}(R[X_1, \cdots, X_n]/P)$$

is smooth. Then for all $x : R^n$ such that $P(x) = 0$ and $k > 1$, we have that:

$$N_{k-1}(P) : N_{k-1}(x) \to N_{k-1}(0)$$

being zero implies that:

$$N_k(P) : N_k(x) \to N_k(0)$$

is zero as well.

Proof Assume given $x : R^n$ such that $P(x) = 0$, using a translation we can assume $x = 0$. Then:

$$N_{k-1}(P) = 0$$

means that $P = 0$ modulo $(X_1, \cdots, X_n)^k$.

Assume given $\epsilon_1, \cdots, \epsilon_n : R$ such that:

$$(\epsilon_1, \cdots, \epsilon_n)^{k+1} = 0$$

then we consider the lift in:

$$\begin{array}{ccc} R/(\epsilon_1, \cdots, \epsilon_n)^k & \xleftarrow{X_i \mapsto \epsilon_i} & R[X_1, \cdots, X_n]/P \\ \uparrow & \swarrow & \\ R & & \end{array}$$

which gives $\delta_1, \cdots, \delta_n : (\epsilon_1, \cdots, \epsilon_n)^k$ such that:

$$P(\epsilon_1 + \delta_1, \cdots, \epsilon_n + \delta_n) = 0$$

Since $P = 0$ modulo $(X_1, \cdots, X_n)^k$, we have:

$$P = \sum_{i_1, \cdots, i_k} c_{i_1, \cdots, i_k} X_{i_1} \cdots X_{i_k} + Q$$

where $Q = 0$ modulo $(X_1, \cdots, X_n)^{k+1}$. By computation we get:

$$\sum_{i_1, \cdots, i_k} c_{i_1, \cdots, i_k} \epsilon_{i_1} \cdots \epsilon_{i_k} = 0$$

So we know that:

$$(\epsilon_1, \cdots, \epsilon_n)^{k+1} = 0$$

implies:

$$\sum_{i_1, \cdots, i_k} c_{i_1, \cdots, i_k} \epsilon_{i_1} \cdots \epsilon_{i_k} = 0$$

so by sqc we can conclude that all $c_{i_1, \cdots, i_k}$ are zeros, so that $P = 0$ modulo $(X_1, \cdots, X_n)^{k+1}$, which indeed means $N_k(P) = 0$. □

Lemma 6.5.3 Assume given $P : R[X_1, \cdots, X_n]$ such that:

$$\mathrm{Spec}(R[X_1, \cdots, X_n]/P)$$

is smooth and $P \neq 0$. Then for all $x : R^n$ the jacobian:

$$J(P)(x) : R^n \to R$$

is surjective.

Proof Since the jacobian is linear and takes value in R, it is enough to prove that it is not equal to zero to conclude that it is surjective. But we have an equivalence:

$$(N_1(x) \to N_1(0)) \simeq \mathrm{Hom}_R(T_x(R^n), T_0(R))$$

sending $N_1(P)$ to $J(P)(x)$ so it is enough to prove that $N_1(P) \neq 0$.

Assume $N_1(P) = 0$, then by lemma 6.5.2 we have $N_k(P) = 0$ for all k and then $P = 0$, which is a contradiction. □

Remark 6.5.4 Expecting a full converse is unreasonable, see for example:

$$\mathrm{Spec}(R[X]/(X - a)(X - b), (X - a)(X - c))$$

which is the point whenever $b \neq c$, so that it is smooth, but has its jacobian going from R to R^2 so never surjective.

6.6 Smooth schemes have free tangent spaces

Lemma 6.6.1 Assume X is a smooth scheme. Then for any $x : X$ the type $T_x(X)$ is formally smooth.

Proof Consider $T(X) = X^{\mathbb{D}(1)}$ the total tangent bundle of X. We have to prove that the map:

$$p : T(X) \to X$$

is formally smooth. Both source and target are schemes, and the source is formally smooth because X is smooth and $\mathbb{D}(1)$ has choice. So by corollary 8.4.2 it is enough to prove that for all $x : X$ and $v : T_x(X)$ the induced map:

$$dp : T_{(x,v)}(T(X)) \to T_x(X)$$

is surjective.

Consider $v' : T_x(X)$. By unpacking the definition of tangent spaces, we see that merely finding $w : T_{(x,v)}(T(X))$ such that $dp(w) = v'$ means merely finding:

$$\phi : \mathbb{D}(1) \times \mathbb{D}(1) \to X$$

such that for all $t : \mathbb{D}(1)$ we have that:

$$\phi(0,t) = v(t)$$
$$\phi(t,0) = v'(t)$$

But we know that there exists a unique:

$$\psi : \mathbb{D}(2) \to X$$

such that:

$$\psi(0,t) = v(t)$$
$$\psi(t,0) = v'(t)$$

used for example to define $(v + w)(t) = \psi(t,t)$.

Then the fact that X is smooth and that:

$$\mathbb{D}(2) \to \mathbb{D}(1) \times \mathbb{D}(1)$$

is a closed dense embedding means that there merely exists a lift of ψ to $\mathbb{D}(1) \times \mathbb{D}(1)$, which gives us the ϕ we wanted. □

Lemma 6.6.2 Assume given a linear map:

$$M : R^m \to R^n$$

which has a formally smooth kernel. Then we can decide whether $M = 0$.

Proof Since $M = 0$ is closed, it is enough to prove that it is $\neg\neg$-stable to conclude that it is decidable. Assume $\neg\neg(M = 0)$, then for any $x : R^m$ we have a dotted lift in:

$$\begin{array}{ccc} M = 0 & \xrightarrow{\_ \mapsto x} & K \\ \downarrow & \nearrow & \\ 1 & & \end{array}$$

because K is formally smooth, so that we merely have $y : K$ such that:

$$M = 0 \to x = y$$

which implies that $\neg\neg(x = y)$ since we assumed $\neg\neg(M = 0)$.

Then considering a basis $(x_1, \cdots, x_n)$ of R^m, we get $(y_1, \cdots, y_n)$ such that for all i we have that $M(y_i) = 0$ and $\neg\neg(y_i = x_i)$. But then we have that $(y_1, \cdots, y_n)$ is infinitesimally close to a basis and that being a basis is an open proposition, so that $(y_1, \cdots, y_n)$ is a basis and $M = 0$. □

Lemma 6.6.3 Assume that K is a finitely copresented module that is also formally smooth. Then it is finite free.

Proof Assume a finite copresentation:

$$0 \to K \to R^m \overset{M}{\to} R^n$$

We proceed by induction on m. By lemma 6.6.2 we can decide whether $M = 0$ or not.

- If $M = 0$ then $K = R^m$ and we can conclude.
- If $M \neq 0$ then we can find a non-zero coefficient in the matrix corresponding to M, and so up to base change it is of the form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \widetilde{M} & \\ 0 & & & \end{pmatrix}$$

  But then we know that the kernel of M is equivalent to the kernel of $\widetilde{M}$, and by applying the induction hypothesis we can conclude that it is finite free. □

Proposition 6.6.4 Let X be a smooth scheme. Then for any $x : X$ we have that $T_x(X)$ is finite free.

Proof By lemma 6.6.1 we have that $T_x(X)$ is formally smooth, so that we can conclude by lemma 6.6.3.□

The dimension of $T_x(X)$ is called the dimension of X at x.

Corollary 6.6.5 Any smooth scheme is a finite disjoint union of component of fixed dimension.

6.7 Smooth schemes are locally standard

Definition 6.7.1 A standard smooth scheme is an affine scheme of the form:

$$\mathrm{Spec}\left((R[X_1, \cdots, X_n, Y_1, \cdots Y_k]/P_1, \cdots, P_n)_G\right)$$

where G divides the determinant of:

$$\left(\frac{\partial P_i}{\partial X_j}\right)_{1 \leq i,j \leq n}$$

in:

$$R[X_1, \cdots, X_n, Y_1, \cdots Y_k]/P_1, \cdots, P_n$$

Lemma 6.7.2 For any standard smooth scheme:

$$\mathrm{Spec}\left((R[X_1, \cdots, X_n, Y_1, \cdots Y_k]/P_1, \cdots, P_n)_G\right)$$

the map:

$$\mathrm{Spec}\left((R[X_1, \cdots, X_n, Y_1, \cdots Y_k]/P_1, \cdots, P_n)_G\right) \to \mathrm{Spec}(R[Y_1, \cdots Y_k])$$

is formally étale.

Proof The fibers of this map are standard étale, so we can conclude by lemma 4.5.3. □

Corollary 6.7.3 A standard smooth scheme:

$$\mathrm{Spec}((R[X_1, \cdots, X_n, Y_1, \cdots Y_k]/P_1, \cdots, P_n)_G)$$

is smooth of dimension k.

Proposition 6.7.4 A scheme is smooth if and only if it has a Zariski cover by standard smooth schemes.

Proof We can assume the scheme X affine, say of the form:

$$X = \mathrm{Spec}(R[X_1, \cdots, X_m]/P_1, \cdots, P_l)$$

By proposition 6.6.4, for any $x : X$ we have that dP_x has free kernel. We partition by the dimension n of the kernel. Then by lemma 10.1.4 we know that dP_x has rank n for every x.

We cover X according to which n-minor is invertible, so that up to a rearranging of variables and polynomials we can assume that:

$$X = \mathrm{Spec}(R[X_1, \cdots, X_n, Y_1, \cdots, Y_k]/P_1, \cdots, P_n, Q_1, \cdots, Q_l)$$

where we have:

$$dP_{x,y} = \begin{pmatrix} \left(\frac{\partial P}{\partial X}\right)_{x,y} & \left(\frac{\partial P}{\partial Y}\right)_{x,y} \\ \left(\frac{\partial Q}{\partial X}\right)_{x,y} & \left(\frac{\partial Q}{\partial Y}\right)_{x,y} \end{pmatrix}$$

where we used the notation:

$$\left(\frac{\partial P}{\partial X}\right)_{x,y} = \left(\left(\frac{\partial P_i}{\partial X_j}\right)_{x,y}\right)_{i,j}$$

so that $\frac{\partial P}{\partial X}$ is invertible of size n. Moreover by lemma 10.1.2 we get:

$$\left(\frac{\partial Q}{\partial Y}\right)_{x,y} = \left(\frac{\partial Q}{\partial X}\right)_{x,y} \left(\frac{\partial P}{\partial X}\right)^{-1}_{x,y} \left(\frac{\partial P}{\partial Y}\right)_{x,y}$$

which will be useful later.

Now we prove that for any $(x, y) : R^{n+k}$ such that $P(x, y) = 0$ it is decidable whether

$$Q(x, y) = 0$$

To do this it is enough to prove that:

$$(Q_1(x, y), \cdots, Q_l(x, y))^2 = 0 \to (Q_1(x, y), \cdots, Q_l(x, y)) = 0$$

Assuming $(Q_1(x, y), \cdots, Q_l(x, y))^2 = 0$, by smoothness there is a dotted lifting in:

$$\begin{array}{ccc} R/(Q_1(x,y),\cdots,Q_l(x,y)) & \xleftarrow{(x,y)} & \mathrm{Spec}(R[X_1,\cdots,X_n,Y_1,\cdots,Y_k]/P_1,\cdots,P_n,Q_1,\cdots,Q_l) \\ \uparrow & & \\ R & \overset{(x',y')}{\leftarrow\cdots\cdots} & \end{array}$$

Let us prove that $Q(x, y) = 0$. Indeed we have $(x, y) \sim_1 (x', y')$ so that we have:

$$P(x, y) = P(x', y') + \left(\frac{\partial P}{\partial X}\right)_{x',y'} (x - x') + \left(\frac{\partial P}{\partial Y}\right)_{x',y'} (y - y')$$

$$Q(x, y) = Q(x', y') + \left(\frac{\partial Q}{\partial X}\right)_{x',y'} (x - x') + \left(\frac{\partial Q}{\partial Y}\right)_{x',y'} (y - y')$$

Then we have $P(x, y) = 0$, $P(x', y') = 0$ and $Q(x', y') = 0$. From the first equality we get:

$$x - x' = -\left(\frac{\partial P}{\partial X}\right)^{-1}_{x',y'} \left(\frac{\partial P}{\partial Y}\right)_{x',y'} (y - y')$$

so that from the second we get:

$$Q(x, y) = -\left(\frac{\partial Q}{\partial X}\right)_{x',y'} \left(\frac{\partial P}{\partial X}\right)^{-1}_{x',y'} \left(\frac{\partial P}{\partial Y}\right)_{x',y'} (y - y') + \left(\frac{\partial Q}{\partial Y}\right)_{x',y'} (y - y')$$

so that $Q(x, y) = 0$ as we have seen previously that:

$$\left(\frac{\partial Q}{\partial Y}\right)_{x',y'} = \left(\frac{\partial Q}{\partial X}\right)_{x',y'} \left(\frac{\partial P}{\partial X}\right)^{-1}_{x',y'} \left(\frac{\partial P}{\partial Y}\right)_{x',y'}$$

From the decidability of $Q(x,y)=0$ we get that X is an open in:

$$\mathrm{Spec}(R[X_1,\cdots,X_n,Y_1,\cdots,Y_k]/P_1,\cdots,P_n)$$

and from there we conclude exactly as in proposition 4.5.4. $\square$

The following definition is reasonable from a synthetic differential geometry standpoint:

Definition 6.7.5 A type X is called a manifold of dimension k if there merely is a span:

$$\mathbb{A}^k \xleftarrow{\;p\;} U \xrightarrow{\;q\;} X$$

where p and q are formally étale and q is surjective.

Proposition 6.7.6 Given a scheme X the following are equivalent:

- The scheme X is formally smooth of dimension k.
- The scheme X is a manifold of dimension k.

Proof The fact that k-manifolds are smooth is immediate, as in the span:

$$\mathbb{A}^k \xleftarrow{\;p\;} U \xrightarrow{\;q\;} X$$

we have that $\mathbb{A}^k$ smooth and p étale implies U smooth by lemma 6.3.2, and then q surjective implies X smooth by proposition 6.3.3. To know that X is of dimension k we use that any étale map induces isomorphisms of tangent spaces plus surjectivity.

For the converse we use proposition 6.7.4 and lemma 6.7.2. $\square$

7 Étale replacement of schemes

7.1 Etale replacement of schemes

We write Cld for the type of closed dense proposition, i.e. closed propositions P such that $\neg\neg P$.

Lemma 7.1.1 Closed dense propositions are closed under Σ.

Proof We know that closed proposition are closed under Σ, so we just need to check that when $\neg\neg P$ and for all $x : P$ we have $\neg\neg Q(x)$ then $\neg\neg(\Sigma_{x:P} Q(x))$. This is easy. $\square$

Lemma 7.1.2 The formally étale replacement of a proposition P is:

$$\exists(Q : \mathrm{Cld}).P^Q$$

Proof First we check that:

$$\exists(Q : \mathrm{Cld}).P^Q$$

is étale. Assume R closed dense such that:

$$R \to \exists(Q : \mathrm{Cld}).P^Q$$

Then since closed propositions satisfy choice and we try to prove a proposition, we can assume:

$$R \to \Sigma_{Q:\mathrm{Cld}} P^Q$$

i.e. we have $Q : R \to \mathrm{Cld}$ such that:

$$\prod_{x:R} P^{Q(x)}$$

Then $\Sigma_{x:R}Q(x)$ is closed dense and it implies P so:

$$\exists(Q : \mathrm{Cld}).P^Q$$

holds.

We know that the formally étale replacement of a proposition is a proposition, so to conclude it is enough to prove that:

$$\exists(Q : \mathrm{Cld}).P^Q$$

is initial among formally étale propositions implied by P. Assume U a formally étale proposition such that $P \to U$, we want to prove that:

$$(\exists(Q : \mathrm{Cld}).P^Q) \to U$$

Since we want to prove a proposition we can assume Q closed dense such that P^Q, then U^Q holds and this implies U since U is formally étale. □

This can be generalised to sets:

Proposition 7.1.3 Let X be a formally unramified set, then the formally étale replacement of X is:

$$\mathrm{colim}_{Q:\mathrm{Cld}} X^Q$$

Proof For any closed dense proposition P and Q such that $P \to Q$, we have that the fibers of:

$$P \to Q$$

are simply P so that this is a closed dense embedding and the map:

$$X^Q \to X^P$$

is an embedding as X is unramified. So we have that:

$$\mathrm{colim}_{Q:\mathrm{Cld}} X^Q$$

is a filtered colimit of embeddings. Now we have the following:

- The colimit is formally unramified, i.e. its identity types are étale. Since it is a filtered colimit of embeddings, it is enough to prove that identity types in X^Q are étale for any closed dense Q, which holds because X is unramified.
- The colimit is formally smooth. By proposition 6.3.3 it is enough to show that:

  $$\Sigma_{Q:\mathrm{Cld}} X^Q$$

  is formally smooth. But this follow from stability of closed dense propositions by dependent sum.
- Now we show that the map:

  $$X \to \mathrm{colim}_{Q:\mathrm{Cld}} X^Q$$

  is étale-connected. Since we have a filitered colimit of embeddings, the fiber over $\phi : X^Q$ is simply the type of filler for:

  $$\begin{array}{ccc} Q & \xrightarrow{\phi} & X \\ \downarrow & & \\ 1 & & \end{array}$$

  This type is a proposition since X is unramified, so its formally étale replacement is a proposition and we just need to check that it is inhabited. But under the formally étale modality we can assume Q and then we have a lift. □

It should be noted that the fact, that X is a set, was only used to define the colimit in the previous proposition, which extends readily to all the cases where the colimit can be defined in HoTT.

Proposition 7.1.4 Let X be formally smooth, then the formally unramified replacement of X is formally étale.

The converse does not hold, e.g. by considering an infinitesimal variety.

Proof The map from X to its formally unramified replacement is surjective by proposition 3.1.4, so the formally unramified replacement is formally smooth by proposition 6.3.3. ☐

Now we focus on étale replacement for schemes.

Lemma 7.1.5 Let P be a closed proposition, then the étale replacement of P is $\neg\neg P$.

Proof We have that $\neg\neg P$ is étale because it is $\neg\neg$-stable. It is initial among maps from P to étale types because $P \to \neg\neg P$ is a closed dense embedding. ☐

Lemma 7.1.6 Let P be an identity types in a scheme, then the étale replacement of P is $\neg\neg P$.

Proof An identity type in a scheme is of the form:

$$\Sigma_{x:U} C(x)$$

for U open and $C(x)$ closed for all $x : U$. Then:

$$\neg\neg(\Sigma_{x:U} C(x)) \to \Sigma_{x:U} \neg\neg C(x)$$

because $\neg\neg U \to U$ and we can conclude because $\neg\neg C(x)$ is the étale replacement of $C(x)$. ☐

Corollary 7.1.7 The unramified replacement of a scheme X is the quotient of X by the relation $\neg\neg(x =_X y)$.

Proof By proposition 3.1.4 combined with the previous lemma. ☐

By combining proposition 7.1.3 and corollary 7.1.7 we can compute the étale replacement of any scheme. By proposition 7.1.4 the second step is not necessary for smooth scheme.

Example 7.1.8 The étale replacement of $\mathbb{A}^n$ is $\tilde{R}^n$ where $\tilde{R}$ is the quotient of R by its nilradical.

The étale replacement of $\mathbb{P}^n$ should be $\mathbb{P}^n_{\tilde{R}}$ but I did not make this precise.

8 Formally smooth, étale and unramified maps between sets

8.1 Neighborhoods for sets

Definition 8.1.1 Let X be a set with $x : X$. The n-order neighborhood $N_n(x)$ is defined as the set of $y : X$ such that the exists a f.g. ideal I such that $I^{n+1} = 0$ and:

$$I = 0 \to x = y$$

We write:

$$N_\infty(x) = \cup_{n:\mathbb{N}} N_n(x)$$

Lemma 8.1.2 For any set X and x, we have that:

$$N_\infty(x) \simeq \sum_{y:Y} \mathrm{Et}(x = y)$$

Proof By lemma 7.1.2. ☐

8.2 Recap on modules and infinitesimal disks

Lemma 8.2.1 For any finitely presented R-module M, we have a natural iso:

$$T_0\mathbb{D}(M) = M^\star$$

Proof This is a direct application of lemma 1.1.15. ☐

Lemma 8.2.2 A linear map between finitely copresented module:

$$f : M \to N$$

is injective (resp. an iso) iff the corresponding pointed map:

$$\mathbb{D}(M^\star) \to \mathbb{D}(N^\star)$$

is an embedding (resp. an equivalence).

Proof The case of isomorphisms is a direct consequence of lemma 1.1.15.

Assume f injective, then $f^\star$ is surjective the induced map:

$$R \oplus N^\star \to R \oplus M^\star$$

is surjective as well, giving an embedding of affine schemes.

Conversely assume the pointed map:

$$\mathbb{D}(M^\star) \to \mathbb{D}(N^\star)$$

is an embedding then the induced map:

$$T_0\mathbb{D}(M^\star) \to T_0\mathbb{D}(N^\star)$$

is a injective as well, but it is equivalent to the map:

$$M \to N$$

by lemma 8.2.1. □

Lemma 8.2.3 A linear map between finitely copresented module:

$$f : M \to N$$

is surjective if and only if the corresponding pointed map:

$$\mathbb{D}(M^\star) \to \mathbb{D}(N^\star)$$

merely has a section preserving 0.

Proof We know that lemma 1.1.15 we know that:

$$\mathbb{D}(M^\star) \to \mathbb{D}(N^\star)$$

merely having a section preserving 0 is equivalent to:

$$M \to N$$

merely having a section. But since any finitely copresented module is projective, this is equivalent to f being surjective. □

8.3 Formally unramified maps between sets

Proposition 8.3.1 Let $f : X \to Y$ be a map between sets. The following are equivalent:

(i) The map f is formally unramified.

(ii) For all $x : X$, the induced map:

$$N_\infty(f) : N_\infty(x) \to N_\infty(f(x))$$

is an embedding.

(iii) For all $x : X$, the induced map:

$$N_1(f) : N_1(x) \to N_1(f(x))$$

is an embedding.

Proof Let us assume (i) to prove (ii). Assume given x with $y, z \in N_\infty(x)$ such that $f(y) = f(z)$, we want to prove $y = z$. But we have $\mathrm{Et}(y = z)$ so there is a closed dense P such that $P \to y = z$, and then y and z are both lifting of the same square:

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ 1 & \longrightarrow & Y \end{array}$$

so they are equal by (i).

Let us assume (ii) to prove (iii). Assume given $x : X$ and $y, z \in N_1(x)$ such that $f(y) = f(z)$. Then $y, z \in N_\infty(x)$ so by (ii) they are equal.

Let us assume (iii) to prove (i). Assume given two lifts $x, y : X$ to a square:

$$\begin{array}{ccc} \epsilon = 0 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ 1 & \longrightarrow & Y \end{array}$$

with $\epsilon^2 = 0$. Then $\epsilon = 0 \to x = y$ so that $x \in N_1(y)$ and since $f(x) = f(y)$, we conclude by (iii) that we have $x = y$. □

Corollary 8.3.2 Let $f : X \to Y$ be a map between schemes. Then the following are equivalent:

(i) The map f is unramified.

(ii) For all $x : X$, the induced map:
$$df : T_x(X) \to T_{f(x)}(Y)$$
is injective.

Proof By proposition 8.3.1 with lemma 8.2.2 □

8.4 Formally smooth maps between sets

Here we do not find an equivalence, but just an implication. This might be possible to correct.

Proposition 8.4.1 Let $f : X \to Y$ be a map between sets. Assume that X is formally smooth and that for all $x : X$, the induced map:
$$N_1(f) : N_1(x) \to N_1(f(x))$$
merely has a section sending $f(x)$ to x. Then f is formally smooth.

Proof Assume given $\epsilon : R$ such that $\epsilon^2 = 0$ and try to merely find a lift in:

$$\begin{array}{ccc} \epsilon = 0 & \xrightarrow{\phi} & X \\ \downarrow & \nearrow & \downarrow f \\ 1 & \xrightarrow{y} & Y \end{array}$$

Since X is formally smooth we merely have an $x : X$ such that:

$$\prod_{p : \epsilon = 0} \phi(p) = x$$

and therefore:
$$\epsilon = 0 \to y = f(x)$$

This means that we can factor the square:

$$\begin{array}{ccccc} \epsilon = 0 & \xrightarrow{\phi} & N_1(x) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow & & \downarrow f \\ 1 & \xrightarrow{y} & N_1(f(x)) & \longrightarrow & Y \end{array}$$

where we can find a lift because the middle arrow has a section sending $f(x)$ to x. □

Note that it is immediate from the definition of smoothness that smooth maps induce surjections on tangent spaces. We have a converse when the domain is smooth.

Corollary 8.4.2 Let $f : X \to Y$ be a map between schemes with X smooth. Then the following are equivalent:

(i) The map f is smooth.

(ii) For all $x : X$, the induced map:

$$df : T_x(X) \to T_{f(x)}(Y)$$

is surjective.

Proof It is straightforward to prove that (i) implies (ii), even without any assumption on X and Y, by considering the diagram:

$$\begin{array}{ccc} 1 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{D}(1) & \longrightarrow & Y \end{array}$$

To prove that (ii) implies (i) we use proposition 8.4.1 with lemma 8.2.3. □

8.5 Formally étale maps between sets

Proposition 8.5.1 Let $f : X \to Y$ be a map between sets. Assume that X is formally smooth. Then the following are equivalent:

(i) The map f is formally étale.

(ii) For all $x : X$, the induced map:

$$N_\infty(f) : N_\infty(x) \to N_\infty(f(x))$$

is an equivalence.

(iii) For all $x : X$, the induced map:

$$N_1(f) : N_1(x) \to N_1(f(x))$$

is an equivalence.

Proof Assume (i) to prove (ii). This is a general propriety of lex modalities that modal maps induces equivalence of the modal disks.

Assume (ii) to prove (iii). For all $x : X$, the inverse to the map:

$$N_\infty(f) : N_\infty(x) \to N_\infty(f(x))$$

has to preserve first-order neighbourhood so it induces an equivalence as in (iii).

Finally we assume (iii) to prove (i). By proposition 8.3.1 we already know that f is unramified. We see that it is smooth by using proposition 8.4.1. □

Corollary 8.5.2 Let $f : X \to Y$ be a map between schemes. Assume X is smooth. Then the following are equivalent:

(i) The map f is étale.

(ii) For all $x : X$, the induced map:

$$df : T_x(X) \to T_{f(x)}(Y)$$

is an iso.

Proof By proposition 8.5.1 with lemma 8.2.2. □

9 Differential forms and de Rham cohomology

9.1 Differential forms

Definition 9.1.1 For any type X, the type $\Delta^k(X)$ of infinitesimal k-simplices in X is defined by:

$$\Delta^k(X) = \{x_0, \cdots, x_n : X \mid \forall i, j.\ x_i \sim_1 x_j\}$$

Definition 9.1.2 An infinitesimal k-simplex $x_0, \cdots, x_n$ in X is called degenerate if there merely exists $i \neq j$ such that $x_i = x_j$.

Definition 9.1.3 For any type X, the type $\Omega^k(X)$ of differential k-forms on X consists of maps:

$$\omega : \Delta^k(X) \to R$$

which are zero on degenerate infinitesimal k-simplices.

9.2 Differential forms are alternating

Lemma 9.2.1 For any f.p. R-algebra A, we have that $\Delta^1(\mathrm{Spec}(A))$ is the type of $x, y : \mathrm{Spec}(A)$ such that for all $f, g : A$ we have:

$$(f(x) - f(y))(g(x) - g(y))$$

Proof We just need to prove that for $x, y : \mathrm{Spec}(A)$, the following are equivalent:

- We have that $x \sim_1 y$.
- For all $f, g : A$ we have:
$$(f(x) - f(y))(g(x) - g(y)) = 0$$

If $x \sim_1 y$ then there is a f.g. ideal I such that $I^2 = 0$ and $I = 0 \to x = y$. Then $I = 0$ implies $f(x) - f(y) = 0$ and $g(x) - g(y) = 0$ so that both belong to I and their product is zero.

Conversely we consider $f_1, \cdots, f_n$ generating A, then we consider the f.g. ideal:

$$(f_1(x) - f_1(y), \cdots, f_n(x) - f_n(y))$$

By hypothesis this ideal has square zero, and if it is null then $x = y$ so we indeed have that $x \sim_1 y$. □

This can surely be extended to k-simplices, although it is unpleasant to write down.

Corollary 9.2.2 For any f.p. R-algebra A, we have that $\Delta^1(\mathrm{Spec}(A))$ is the spectrum of $A \otimes A$ quotiented by:

$$(f \otimes 1 - 1 \otimes f)(g \otimes 1 - 1 \otimes g)$$

for all $f, g : A$. If $f_1, \cdots, f_n$ generate A, it is enough to quotient by:

$$(f_i \otimes 1 - 1 \otimes f_i)(f_j \otimes 1 - 1 \otimes f_j)$$

for all i, j.

Proof The first part is just sqc. In the finitely generated case we check that:

$$(ff' \otimes 1 - 1 \otimes ff')(g \otimes 1 - 1 \otimes g)$$

belongs to the ideal generated by:

$$(f \otimes 1 - 1 \otimes f)(g \otimes 1 - 1 \otimes g)$$

and

$$(f' \otimes 1 - 1 \otimes f')(g \otimes 1 - 1 \otimes g)$$

□

Lemma 9.2.3 For any $\omega : \Omega^1(X)$ and $(x, y) : \Delta^1(X)$ we have that:

$$\omega(x, y) = -\omega(y, x)$$

Proof By corollary 9.2.2 we know that there exists some $f_i, g_i : A$ such that for all $(x, y) : \Delta^1(X)$ we have that:

$$\omega(x, y) = \sum_i f_i(x) g_i(y)$$

Then we need to check that for any $(x, y) : \Delta^1(X)$ we have:

$$\omega(x, y) = -\omega(y, x)$$

But by lemma 9.2.1, for any i we have that:

$$(f_i(x) - f_i(y))(g_i(x) - g_i(y)) = 0$$

so that we have that:

$$\sum_i f_i(x)g_i(x) - \sum_i f_i(x)g_i(y) - \sum_i f_i(y)g_i(x) + \sum_i f_i(y)g_i(y) = 0$$

but since for any $x : \mathrm{Spec}(A)$ we have:

$$\sum_i f_i(x)g_i(x) = \omega(x, x) = 0$$

We can conclude that:

$$\sum_i f_i(x)g_i(y) + \sum_i f_i(y)g_i(x) = 0$$

which is what we want. □

Proposition 9.2.4 Any $\omega : \Omega^k(X)$ is alternating, meaning that for any $x = (x_0, \cdots, x_n) : \Delta^k(X)$ and σ is a permutation of $n + 1$ elements, we have that:

$$\omega(\sigma x) = \mathrm{sign}(\sigma)\omega(x)$$

Proof It is enough to show this for the exchange x_i and x_j. But by fixing x_k for $k \neq i, j$, this is lemma 9.2.3 applied in the intersection of the first-order neighbourhood of x_k for $k \neq i, j$. □

9.3 Differential forms and the de Rham complex

TODO in low dimension...

10 Miscellaneous linear algebra

We present some sketches of synthetic linear algebra.

Lemma 10.0.1 Let X be an infinitesimal variety. Then choice over X is valid: for any type family P with $(x : X) \to \|P(x)\|$, we have $\|(x : X) \to P(x)\|$.

Proof By Zariski local choice, it suffices to show that every Zariski cover of $X \to R$ is trivial, that is that $X \to R$ is a local ring. Indeed this is the case, since the evaluation $(X \to R) \to R$ reflects invertible elements. □

Lemma 10.0.2 In the category of finitely co-presented R-modules, every object is projective. That is, if M, N, L are finitely co-presented R-modules, $f : M \to L$ is R-linear and $g : N \to L$ is R-linear and surjective, then there merely exists $h : M \to N$ such that $f = g \circ h$.

Proof We apply lemma 10.0.1 to obtain $(m : \mathbb{D}(M^\star)) \to (n : N) \times g(n) = f(m)$. That is, we have $h_0 : \mathbb{D}(M^\star) \to N$ such that $f = g \circ h_0$ on $\mathbb{D}(M)$. Without loss of generality, $h_0(0) = 0$; otherwise we may replace h_0 by $h_0 - h_0(0)$, since in any case $g(h_0(0)) = f(0) = 0$. Now h_0 lifts to a pointed map $\mathbb{D}(M^\star) \to_{\mathrm{pt}} \mathbb{D}(N^\star)$. This corresponds to an R-linear map $N^\star \to M^\star$, and hence to an R-linear map $M \to N$, as desired. □

In particular, since R is a local ring, we have that if M is finitely co-presented and finitely generated, then M is free.

Note that for any R-linear map $f : M \to N$, we have $(\text{coker}(f))^\star = \ker(f^\star)$, where $f^\star : N^\star \to M^\star$ is the dual map. If M and N are finitely presented, then so is $\text{coker}(f)$, and $\ker(f^\star)$ is finitely co-presented. Hence if M and N are finitely presented, then f is surjective iff $f^\star$ is injective.

Definition 10.0.3 For M finitely presented, we say $\dim M \leq n$ if there merely exists a surjective linear map $R^n \to M$.

For V finitely co-presented, we say $\dim V \leq n$ if there merely exists an injective linear map $V \to R^n$.

It is direct that there merely exists n such that $\dim M \leq n$ in either case. Note that as usual in constructive algebra, we do not define $\dim M$ as a natural number, but only what it means to compare it with natural numbers. To see that our notation is consistent, consider M which is both finitely presented and finitely co-presented. In this case M is free of some rank k, and we have that $\dim M = k$, in the appropriate sense.

Lemma 10.0.4 We have $\dim M \leq n$ iff $\dim M^\star \leq n$.

Proof Follows from $(R^n)^\star = R^n$ and the fact that f is surjective if and only if $f^\star$ is injective. □

Lemma 10.0.5 A map $f : R^m \to R^n$ is surjective if and only if the induced map $\bigwedge^n f : \bigwedge^n R^m \to \bigwedge^n R^n$ on the nth exterior power is non-zero. In particular, this is an open proposition, asserting that a certain list of $\binom{m}{n}$ numbers is nonzero.

Proof For the forward implication, pick preimages for the basis vectors, $f(u_i) = e_i$, and note $f(u_1 \wedge \ldots \wedge u_n) = e_1 \wedge \ldots \wedge e_n \neq 0$. For the reverse implication, if $\bigwedge^n f$ is nonzero, then there are $u_1, \ldots, u_n : R^m$ such that $f(u_1 \wedge \ldots \wedge u_n) = e_1 \wedge \ldots \wedge e_n$. This means that u_i determine a map $R^n \to R^m$ such that the composite $R^n \to R^m \to R^n$ has invertible determinant and hence is surjective. □

Lemma 10.0.6 For any linear map $f : R^m \to R^n$, it is not not the case that f is a composite $R^m \simeq R^r \oplus R^{m-r} \to R^r \to R^r \oplus R^n \simeq R^n$, where $r \leq m, n$, the maps to and from R^r are projections and inclusions from and to a direct sum, and the outer isomorphisms are arbitrary.

Proof Since we are proving a negated statement, we can pretend that R is a discrete field. In this case we follow a well-known matrix algorithm. □

In particular this shows that any finitely presented module is not not free.

Lemma 10.0.7 $\dim M \leq n$ is an open proposition.

Proof Let M be represented as the cokernel of a map $f : R^k \to R^l$. We claim that $\dim M \leq n$ is equivalent to the assertion that for some detachable subset $I \subseteq [l]$ of size at most n, the composite $R^I \to R^l \to M$ is surjective. Surjectivity of this map is equivalent to surjectivity of $R^I \oplus R^k \to R^l$, which is an open proposition by lemma 10.0.5. Since open propositions are closed under finite disjunction, it is enough to prove our claim. One direction is clear: if $R^I \to M$ is surjective, then $\dim M \leq |I| \leq n$. Conversely, suppose $\dim M \leq n$. Since our goal is to prove an open proposition, and M is not not free, we may assume M is free of rank $r \leq n$. In this case, since we have a surection $R^l \to R^r$, we obtain a subset $I \subseteq [l]$ of size r such that the composite $R^I \to R^l \to R^r$ is surjective, as needed. □

From this proof, it is natural to define $\dim M \geq n$ as follows. If M is the cokernel of $f : R^k \to R^l$, then $\dim M \geq n+1$ iff $\bigwedge^l (\iota_I \oplus f) : \bigwedge^l (R^I \oplus R^k) \to \bigwedge^l R^l$ is zero for all $I \subseteq [l]$ of size less than n, with $\iota_I : R^I \to R^l$ the usual inclusion. This would make $\dim M \geq n$ a closed proposition, whose negation is $\dim M \leq n-1$. But with this definition, it is unclear what $\dim M \geq n$ means for M directly. For example, it does not imply that there exists a surjection $M \to R^n$, even if $k = l = 2$ and $n = 1$.

Definition 10.0.8 For X a scheme, we take $\dim X \leq n$ to mean that the set of $p : X$ with $\dim T_pX \leq n$ is dense in X.

10.1 Rank of matrices

This section should probably go elsewhere.

Definition 10.1.1 A matrix is said of rank n if it has an invertible n-minor, and all its $n+1$-minor have determinant 0.

Beware that having a rank is a property of matrices, and there is not rank function defined on all matrices.

Lemma 10.1.2 Assume given a matrix M of rank n decomposed into blocks:

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

Such that P is square of size n and invertible. Then we have:

$$S = RP^{-1}Q$$

Proof TODO □

Definition 10.1.3 Two matrices M, N are said equivalent if there are invertible matrices P, Q such that $M = PNQ$.

It is clear that equivalent matrices have the same rank.

Lemma 10.1.4 Assume given a matrix:

$$M : R^m \to R^k$$

Then the following are equivalent:

(i) M has rank n.

(ii) The kernel of M is equivalent to R^{m-n}.

(iii) The image of M is equivalent to R^n.

(iv) M is equivalent to the bloc matrix:

$$\begin{pmatrix} I_n & (0) \\ (0) & (0) \end{pmatrix}$$

Proof TODO □

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