Differential Geometry of Synthetic Schemes (toward article)

September 24, 2024

Trying to write down something publishable from the draft on differential geometry.
By Felix Cherubini, Matthias Hutzler, Hugo Moeneclaey and David Wärn.

Contents

1 Formally étale, unramified and smooth types

1.1 Definition

Definition 1.1.1 A closed proposition is dense if it is merely of the form:

$$r_1 = 0 \wedge \cdots \wedge r_n = 0$$

for $r_1, \cdots, r_n : R$ nilpotent.

Definition 1.1.2 A type X is formally étale (resp. formally unramified, formally smooth) if for all closed dense proposition P the map:

$$X \to X^P$$

is an equivalence (resp. an embedding, surjective).

Remark 1.1.3 The map $X \to X^P$ being an equivalence (resp. an embedding, surjective) if and only if for any map $P \to X$ we have a unique (resp. at most one, merely one) dotted lift in:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \\ 1 \arrow[ur, dotted] & \end{tikzcd}$$

Definition 1.1.4 A map is said formally étale (resp. formally unramified, formally smooth) if its fibers are formally étale (resp. formally unramified, formally smooth).

Remark 1.1.5 A type (or map) is formally étale if and only if it is formally unramified and formally smooth.

Lemma 1.1.6 A type X is formally étale (resp. formally unramified, formally smooth) if and only if for all $\epsilon : R$ such that $\epsilon^2 = 0$, the map:

$$X \to X^{\epsilon=0}$$

is an equivalence (resp. an embedding, surjective).

Proof The direct direction is obvious as $\epsilon = 0$ is closed dense when $\epsilon^2 = 0$.

For the converse, assume $P = \mathrm{Spec}(R/N)$ a closed dense proposition. Then the map $R \to R/N$ with N f.g. nilpotent ideal can be decomposed as:

$$R \to A_1 \to \cdots A_n = R/N$$

where A_k is a quotient of R by a f.g. nilpotent ideal and:

$$A_k \to A_{k+1}$$

is of the form:

$$A \to A/(a)$$

for some $a : A$ with $a^2 = 0$.

We write $P_k = \mathrm{Spec}(A_k)$ and:

$$i_k : P_{k+1} \to P_k$$

so that $\mathrm{fib}_{i_k}(x)$ is $a(x) = 0$ where $a(x)^2 = 0$ holds.

Then by hypothesis we have that for all k and $x : P_k$ the map:

$$X \to X^{\mathrm{fib}_{i_k}(x)}$$

is an equivalence (resp. an embedding, surjective). So the map:

$$X^{P_k} \to \prod_{x:P_k} X^{\mathrm{fib}_{i_k}(x)} = X^{P_{k+1}}$$

is an equivalence (resp. an embedding, surjective by P_k having choice). We conclude that the map:

$$X \to X^P$$

is an equivalence (resp. an embedding, surjective). □

1.2 Stability results

Being formally étale is a modality given as nullification at all dense closed propostions and therefore lex [RSS20][Corollary 3.12]. This means we have the following results:

Proposition 1.2.1

- If X is any type and for all $x : X$ we have a formally étale type Y_x, then:

$$\prod_{x:X} Y_x$$

is formally étale.

- If X is formally étale and for all $x : X$ we have a formally étale type Y_x, then:

$$\sum_{x:X} Y_x$$

is formally étale.

- If X is formally étale then for all $x, y : X$ the type $x = y$ is formally étale.
- The type of formally étale types is formally étale.

Formally unramified type are the separated types [Chr+20][Definition 2.13] associated to formally étale types. By [Chr+20][Lemma 2.15], being formally unramified is a nullification modality as well.

Lemma 1.2.2 A type X is formally unramified if and only if for any $x, y : X$ the type $x = y$ is formally étale.

This means we have the following:

Proposition 1.2.3
- If X is any type and for all $x : X$ we have a formally unramified type Y_x, then:
$$\prod_{x:X} Y_x$$
is formally unramified.
- If X is formally unramified and for all $x : X$ we have a formally unramified type Y_x, then:
$$\sum_{x:X} Y_x$$
is formally unramified.

Being formally smooth is not a modality, indeed we will see it is not stable under identity types. Neverthless we have the following results:

Lemma 1.2.4
- If X is any type satifying choice and for all $x : X$ we have a formally smooth type Y_x, then:
$$\prod_{x:X} Y_x$$
is formally smooth.
- If X is a formally smooth type and for all $x : X$ we have a formally smooth type Y_x, then:
$$\sum_{x:X} Y_x$$
is formally smooth.

1.3 Basic examples

Next proposition implies that open propositions are formally étale.

Lemma 1.3.1 Any $\neg\neg$-stable proposition is formally étale.

Proof Assume U is a $\neg\neg$-stable proposition. For U to be formally étale it is enough to check that $U^P \to U$ for all P closed dense. This holds because for P closed dense we have $\neg\neg P$. □

Lemma 1.3.2 A closed and formally étale proposition is decidable.

Proof Given a formally étale closed proposition P, let us prove it is $\neg\neg$-stable. Indeed if $\neg\neg P$ then P is closed dense so that $P \to P$ implies P since P is formally étale.

Let I be the f.g. ideal in R such that:
$$P \leftrightarrow I = 0$$
We have that $I^2 = 0$ implies $\neg\neg(I = 0)$ which implies $I = 0$. But then we have that $I = I^2$, so that by Nakayama (see [LQ15, Lemma II.4.6]) there exists $e : R$ such that $eI = 0$ and $1 - e \in I$. If e is invertible then $I = 0$, if $1 - e$ in invertible then $I = R$. □

Proposition 1.3.3 The type Bool is formally étale.

Proof The identity types in Bool are decidable so Bool is formally unramified. Consider $\epsilon : R$ such that $\epsilon^2 = 0$ and a map:
$$\epsilon = 0 \to \mathrm{Bool}$$
we want to merely factor it through 1.

Since $\mathrm{Bool} \subseteq R$, by duality the map gives $f : R/(\epsilon)$ such that $f^2 = f$. Since $R/(\epsilon)$ is local we conclude that $f = 1$ or $f = 0$ and so the map has constant value $0 : \mathrm{Bool}$ or $1 : \mathrm{Bool}$. □

Remark 1.3.4 This means that formally étale (resp. formally unramified, formally smooth) types are stable by finite sums. In particular finite types are formally étale.

Proposition 1.3.5 The type $\mathbb{N}$ is formally étale.

Proof Identity types in $\mathbb{N}$ are decidable so $\mathbb{N}$ is formally unramified, we want to show it is formally smooth. Assume given a map:

$$P \to \mathbb{N}$$

for P a closed dense proposition, we want to show it merely factors through 1. By boundedness the map merely factors through a finite type, which is formally étale by remark 1.3.4 so we conclude. $\square$

Lemma 1.3.6 Any proposition is formally unramified.

This means that any subtype of a formally unramified type is formally unramified.

Remark 1.3.7 Given any lex modality, a type is separated if and only if it is a subtype of a modal type, so a type is formally unramified if and only if it is a subtype of a formally étale type.

We also have the following surprising dual result, meaning that any quotient of a formally smooth type is formally smooth:

Proposition 1.3.8 If X is formally smooth and $p : X \to Y$ surjective, then Y is formally smooth.

Proof For any P closed dense and any diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, dotted] & Y \\ 1 \arrow[r, dotted, "x"'] & X \arrow[u, "p"'] \end{tikzcd}$$

by choice for closed propositions we merely get the dotted diagonal, and since X is formally smooth we get the dotted x, and then $p(x)$ gives a lift. $\square$

1.4 Being formally étale, unramified or smooth is Zariski local

Lemma 1.4.1 Let X with $(U_i)_{i:I}$ be a finite open cover of X. Then X is formally étale (resp. formally unramified, formally smooth) if and only if all the U_i are formally étale (resp. formally unramified, formally smooth).

Proof First, we show this for formally unramified:

- Any subtype of a formally unramified type is formally unramified by lemma 1.3.6.
- Conversely, assume X with such a cover, for all $x, y : X$ there exists $i : I$ such that $x \in U_i$ and then:

$$x =_X y \leftrightarrow \sum_{y \in U_i} x =_{U_i} y$$

which is formally étale because open propositions are formally étale by lemma 1.3.1.

Now for formally smooth:

- Open proposition are formally smooth by lemma 1.3.1 so that open in a formally smooth are formally smooth.
- Conversely if each U_i is formally smooth then $\Sigma_{i:I} U_i$ is formally smooth by remark 1.3.4, so we can conclude by applying proposition 1.3.8 to the surjection:

$$\Sigma_{i:I} U_i \to X$$

The result for formally étale immediately follows. $\square$

2 Linear algebra and tangent spaces

2.1 Modules and infinitesimal disks

The most basic infinitesimal schemes are the first order neighbourhoods in affine n-space R^n. Their algebra of functions is R^{n+1}, which is an instance of the more general construction below.

For any R-module M, there is an R-algebra structure on $R \oplus M$ with multiplication given by

$$(r, m)(r', m') = (rr', rm' + r'm)$$

Algebras of this form are called *square zero extensions* of R, since products of the form $(0, m)(0, n)$ are zero. By this property, for any R-linear map $\varphi : M \to N$ between modules M, N, the map $\mathrm{id} \oplus \varphi : R \oplus M \to R \oplus N$ is an R-algebra homomorphism. In particular, if M is finitely presented, i.e. merely the cokernel of some $p : R^n \to R^m$ then $R \oplus M$ is the cokernel of a map between finitely presented algebras and therefore finitely presetend as an algebra.

Definition 2.1.1 Given M a finitely presented R-module, we define an f.p. algebra structure on $R \oplus M$ as above and set:

$$\mathbb{D}(M) :\equiv \mathrm{Spec}(R \oplus M)$$

This is a pointed scheme by the first projection which we denote 0 and the construction is functorial by the discussion above.

We write $\mathbb{D}(n)$ for $\mathbb{D}(R^n)$ so that for example:

$$\mathbb{D}(1) = \mathrm{Spec}(R[X]/(X^2)) = \{\epsilon : R \mid \epsilon^2 = 0\}$$

Definition 2.1.2 Assume given M an f.p. R-module and A an f.p. R-algebra with $x : \mathrm{Spec}(A)$. An M-derivation at x is a morphism of R-modules:

$$d : A \to M$$

such that for all $a, b : A$ we have that:

$$d(ab) = a(x)d(b) + b(x)d(a)$$

Lemma 2.1.3 Assume given M an f.p. module and A an f.p. algebra with $x : \mathrm{Spec}(A)$. Pointed maps:

$$\mathbb{D}(M) \to_{\mathrm{pt}} (\mathrm{Spec}(A), x)$$

corresponds to M-derivation at x.

Proof Such a pointed map correponds to an algebra map:

$$f : A \to R \oplus M$$

where the composite with the first projection is x. This means that, for some module map $d : A \to M$ we have:

$$f(a) = (a(x), d(a))$$

We can immediately see that f being a map of R-algebras is equivalent to d being an M-derivation at x. □

Lemma 2.1.4 Let M, N be finitely presented modules. Then linear maps $M \to N$ correspond to pointed maps $\mathbb{D}(N) \to_{\mathrm{pt}} \mathbb{D}(M)$.

Proof By lemma 2.1.3 such a pointed map corresponds to an N-derivation at $0 : \mathbb{D}(M)$.

Such a derivation is a morphism of modules:

$$d : R \oplus M \to N$$

such that for all $(r, m), (r', m') : R \oplus M$ we have that:

$$d(rr', rm' + r'm) = rd(r', m') + r'd(r, m)$$

This implies $d(r, 0) = 0$ for all $r : R$, so we obtain a section to the injective functorial action of $\mathbb{D}$ on linear maps. □

2.2 Tangent spaces

Definition 2.2.1 Let X be a type and let $x : X$, then we define the tangent space $T_x(X)$ of X at x by:

$$\{t : \mathbb{D}(1) \to X \mid t(0) = x\}$$

Definition 2.2.2 Given $f : X \to Y$ and $x : X$ we have a map:

$$df_x : T_x(X) \to T_{f(x)}(Y)$$

induced by post-composition.

Lemma 2.2.3 For all $x : R^n$ we have $T_x(R^n) = R^n$.

Proof Since R^n is homogeneous we can assume $x = 0$. By lemma 2.1.3 we know that $T_0(R^n)$ corresponds to the type of linear maps

$$R[X_1, \cdots, X_n] \to R$$

such that for all P, Q we have:

$$d(PQ) = P(0)dQ + Q(0)dP$$

which is equivalent to $d(1) = 0$ and $d(X_i X_j) = 0$, so any such map is determined by its image on the X_i so it is equivalent to an element of R^n. □

Lemma 2.2.4 Given a scheme X with $x : X$ and $v, w : T_x(X)$, there exists a unique:

$$\psi_{v,w} : \mathbb{D}(2) \to_{\mathrm{pt}} X$$

such that for all $\epsilon : \mathbb{D}(1)$ we have that:

$$\psi_{v,w}(\epsilon, 0) = v(\epsilon)$$
$$\psi_{v,w}(0, \epsilon) = w(\epsilon)$$

Proof We can assume X is affine. Then $\mathbb{D}(2) \to_{\mathrm{pt}} X$ is equivalent to the type of R^2-derivations at x, but giving an $M \oplus N$-derivation is equivalent to giving an M-derivation and an N-derivation. Checking the equalities is a routine computation. □

Lemma 2.2.5 For any scheme X and $x : X$, we have that $T_x(X)$ is a module.

Proof We define scalar multiplication by sending v to $t \mapsto v(rt)$.

Then to define addition of $v, w : T_x(X)$, we have define:

$$(v + w)(\epsilon) = \psi_{v,w}(\epsilon, \epsilon)$$

where $\psi_{v,w}$ is defined in lemma 2.2.4.

We omit checking that this is a module structure. □

Lemma 2.2.6 For $f : X \to Y$ a map between schemes, for all $x : X$ the map df_x is a map of R-modules.

Proof Commutation with scalar multiplication is immediate.

Commutation with addition comes by applying uniqueness in lemma 2.2.4 to show:

$$f \circ \psi_{v,w} = \psi_{f \circ v, f \circ w}$$ □

Lemma 2.2.7 For any map $f : X \to Y$ and $x : X$, we have that:

$$\mathrm{Ker}(df_x) = T_{(x, \mathrm{refl}_{f(x)})}(\mathrm{fib}_f(f(x)))$$

Proof This holds because:

$$(\mathrm{fib}_f(f(x)), (x, \mathrm{refl}_{f(x)}))$$

is the pullback of:

$$(X, x) \to (Y, f(x)) \leftarrow (1, *)$$

in pointed types, applied using $(\mathbb{D}(1), 0)$. □

Lemma 2.2.8 Let X be a scheme with $x : X$. Then $T_x(X)$ is a finitely co-presented R-module.

Proof We cab assume X affine. Then X is the kernel of a map:

$$P : R^m \to R^n$$

so that for all $x : X$, by applying lemma 2.2.7 we know that we have $T_x(X)$ is the kernel of:

$$dP_x : T_x(R^m) \to T_0(R^n)$$

and we conclude by lemma 2.2.3. $\square$

Corollary 2.2.9 Let X be a scheme, then the tangent bundle $X^{\mathbb{D}(1)}$ is a scheme.

Proof We give two independent arguments, the first uses the lemma, the second is a direct computation:

(i) Finitely co-presented modules are schemes, since they are the common zeros of linear functions on R^n. So by the lemma, all tangent spaces $T_x(X)$ are schemes and

$$X^{\mathbb{D}(1)} = \sum_{x:X} T_x(X)$$

is a dependent sum of schemes and therefore a scheme.

(ii) Let X be covered by open affine $U_1, \dots, U_n$ then $U_1^{\mathbb{D}(1)}, \dots, U_n^{\mathbb{D}(1)}$ is an open cover by double negation stability of opens. So we conclude by showing that for affine $Y = \operatorname{Spec} R[X_1, \dots, X_n]/(f_1, \dots, f_l)$ the tangent bundle $Y^{\mathbb{D}(1)}$ is affine by direct computation:

$$\begin{aligned} Y^{\mathbb{D}(1)} &= \operatorname{Hom}_{R\text{-Alg}}(R[X_1, \dots, X_n]/(f_1, \dots, f_l), R \oplus \epsilon R) \\ &= \{(y_1, \dots, y_n) : R \oplus \epsilon R \mid \forall_i . f_i(y_1, \dots, y_n) = 0\} \\ &= \{(x_1, \dots, x_n, d_1, \dots, d_n) : R^{2n} \mid \forall_i . f_i(x_1, \dots, x_n) = 0 \text{ and } \sum_j d_j \frac{\partial f_i}{\partial X_j}(x_1, \dots, x_n) = 0\} \end{aligned}$$ $\square$

2.3 Infinitesimal neighbourhoods

Definition 2.3.1 Let X be a set with $x : X$. The first order neighborhood $N_1(x)$ is defined as the set of $y : X$ such that there exists an f.g. ideal $I \subseteq R$ with $I^2 = 0$ and:

$$I = 0 \to x = y$$

Lemma 2.3.2 Assume $x, y : R^n$, then $x \in N_1(y)$ if and only if the ideal generated by the $x_i - y_i$ has square zero.

Proof Let us denote I the ideal generated by the $x_i - y_i$ so that we clearly have $x = y$ if and only if $I = 0$.

If $I^2 = 0$ then it is clear that $y \in N_1(x)$.

Conversely if $J = 0 \to I = 0$ then we have that $I \subset J$ by duality so that $J^2 = 0$ implies $I^2 = 0$. $\square$

Lemma 2.3.3 Let X be a scheme with $x : X$. Then $N_1(x)$ is an affine scheme.

Proof If $x \in U$ open in X, we have that $N_1(x) \subset U$ so that we can assume X affine.

This means X is a closed subscheme $C \subset R^n$. Then by lemma 2.3.2, we have that $N_1(x)$ is the type of $y : R^n$ such that $y \in C$ and for all i, j we have that $(x_i - y_i)(x_j - y_j) = 0$, which is a closed subset of C so it is an affine scheme. $\square$

Definition 2.3.4 A pointed scheme $(X, *)$ is called a *first order (infinitesimal) disk* if for all $x : X$ we have $x \in N_1(*)$.

Lemma 2.3.5 A pointed scheme $(X, *)$ is a first order disk if and only if there exists a finitely presented module M such that:

$$(X, *) = (\mathbb{D}(M), 0)$$

Proof First we check that for all M finitely presented and $y : \mathbb{D}(M)$ we have that $y \in N_1(0)$. Let $m_1, \cdots, m_n$ be generators of M, then consider $d : M \to R$ induced by y, then $y = 0$ if and only if $d = 0$ and for all i, j we have that:

$$d(m_i)d(m_j) = 0$$

This means that $I = (d(m_1), \cdots, d(m_n))$ has square 0 and $I = 0$ implies $y = 0$ so that $y \in N_1(0)$.

For the converse we assume X a first order disk, by lemma 2.3.3 we have that X is affine and pointed, up to translation we can assume $X \subset R^n$ closed pointed by 0. Since X is a first order disk we have that $X \subset N_1(0)$ and by lemma 2.3.2 we have $N_1(0) = \mathbb{D}(R^n)$.

This means there is an f.g. ideal J in $R \oplus R^n$ such that $X = \mathrm{Spec}(R \oplus R^n/J)$. But $0 \in X$ corresponds to the first projection from $R \oplus R^n$ – meaning if $(x, y) \in J$ then $x = 0$, so that J corresponds uniquely to an f.g. sub-module K of R^n and:

$$X = \mathrm{Spec}(R \oplus (R^n/K)) = \mathbb{D}(R^n/K)$$

□

Lemma 2.3.6 The functor from finitely copresented modules to first order disks:

$$M \mapsto \mathbb{D}(M^\star)$$

is an equivalence, with inverse:

$$(X, x) \mapsto T_x(X)$$

Proof It is fully faithful by lemma 2.1.4 and essentially surjective by lemma 2.3.5. To check for the inverse it is enough to check that:

$$T_0(\mathbb{D}(M^\star)) = M$$

which is also a consequence of lemma 2.1.4. □

Lemma 2.3.7 Let X be a scheme with $x : X$, then we have:

$$N_1(x) = \mathbb{D}(T_x(X)^\star)$$

Proof By lemma 2.3.3 we have that $(N_1(x), x)$ is a first order disk. By lemma 2.3.6 it is enough to check that $T_x(N_1(x)) = T_0(\mathbb{D}(T_x(X)^\star))$.

It is immediate that any map $f : \mathbb{D}(1) \to X$ uniquely factors through $N_1(f(0))$ so that $T_x(N_1(x)) = T_x(X)$, and we have that $T_0(\mathbb{D}(T_x(X)^\star)) = T_x(X)$ by lemma 2.3.6. □

2.4 Projectivity of finitely copresented modules

Lemma 2.4.1 Let M be a finitely copresented module, then we have $T_0(M) = M$.

Proof We have that M is the kernel of a linear map $P : R^m \to R^n$. By lemma 2.2.7 we have that $T_0(M)$ is the kernel of:

$$dP_0 : T_0(R^m) \to T_0(R^n)$$

but by lemma 2.2.3 this is a map from R^m to R^n, we omit the verification that $dP_0 = P$. □

Lemma 2.4.2 Any finitely copresented module is projective.

Proof We consider M, N finitely copresented with a surjective map:

$$f : M \to N$$

By lemma 2.3.7 and lemma 2.4.1 we know that $\mathbb{D}(M^*) = N_1(0)$ in M, so that we have a commutative diagram:

$$\begin{tikzcd}
\mathbb{D}(M^\star) \arrow[r] \arrow[d] & M \arrow[d, "f"] \\
\mathbb{D}(N^\star) \arrow[r, "i"] & N
\end{tikzcd}$$

Since $\mathbb{D}(N^*)$ has choice and f is surjective there is $g : \mathbb{D}(N^\star) \to M$ such that $f \circ g = i$. Up to translation we can assume $g(0) = 0$. Then we can factor g through $\mathbb{D}(M^\star)$ as N_1 is functorial. This gives us a pointed section of the map:

$$\mathbb{D}(M^\star) \to \mathbb{D}(N^\star)$$

which by lemma 2.1.4 gives a linear section of f. (TODO should we check functoriality?) $\square$

Lemma 2.4.3 A linear map between finitely copresented module:

$$f : M \to N$$

is surjective if and only if the corresponding pointed map:

$$\mathbb{D}(M^\star) \to \mathbb{D}(N^\star)$$

merely has a section preserving 0.

Proof By lemma 2.1.4 we know that:

$$\mathbb{D}(M^\star) \to \mathbb{D}(N^\star)$$

merely having a section preserving 0 is equivalent to:

$$f : M \to N$$

merely having a section. But since any finitely copresented module is projective, this is equivalent to f being surjective. $\square$

2.5 Rank of matrices

Definition 2.5.1 A matrix is said of rank n if it has an invertible n-minor, and all its $n+1$-minor have determinant 0.

Having a rank is a property of matrices, as there is no rank function defined on all matrices.

Lemma 2.5.2 Assume given a matrix M of rank n decomposed into blocks:

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

Such that P is square of size n and invertible. Then we have:

$$S = RP^{-1}Q$$

Proof By columns manipulation the matrix is equivalent to:

$$M = \begin{pmatrix} P & 0 \\ 0 & S - RP^{-1}Q \end{pmatrix}$$

but equivalent matrices have the same rank so $S = RP^{-1}Q$. $\square$

Lemma 2.5.3 If a linear map:

$$M : R^m \to R^n$$

has kernel R^k, then it has rank $m - k$.

Proof Let $a_1, \cdots, a_n$ be a basis for the kernel of M in R^m, which we complete into a basis of R^m via $b_{n+1}, \cdots, b_m$. By completing $f(b_{n+1}), \cdots, f(b_m)$ to a basis of R^n, we get a basis where M is written as:

$$\begin{pmatrix} I_n & (0) \\ (0) & (0) \end{pmatrix}$$

so that M has rank n. $\square$

3 Formally unramified schemes

Lemma 3.0.1 Let X be an affine scheme, the following are equivalent:

(i) X is formally unramified.

(ii) Identity types in X are decidable.

(iii) For all $x : X$, we have that $T_x(X) = 0$.

Proof (i) implies (ii). By lemma 1.3.2.

(ii) implies (i). Because decidable implies formally étale.

(ii) implies (iii). Assume given $x : X$ with $t : T_x(X)$, then for all $\epsilon : \mathbb{D}(1)$ we have $\neg\neg(\epsilon = 0)$ so that we have $\neg\neg t(\epsilon) = t(0)$ which implies $t(\epsilon) = t(0)$ since equality is assumed decidable. Therefore $t = 0$ in $T_x(X)$.

(iii) implies (ii). Indeed given $\epsilon : R$ such that $\epsilon^2 = 0$, assume $x, y : X$ such that $\epsilon = 0 \to x = y$. Then $x \in N_1(y)$ and by lemma 2.3.7 and $T_y(X) = 0$ we conclude $x = y$. □

Corollary 3.0.2 Let X be a scheme, the following are equivalent:

(i) X is formally unramified.

(ii) Identity types in X are open.

(iii) For all $x : X$, we have that $T_x(X) = 0$.

Proof Assume $(U_i)_{i:I}$ a finite cover of X by affine schemes. By lemma 1.4.1 we have that X is formally unramified if and only U_i is formally unramified for all $i : I$.

(ii) implies (i). By lemma 1.3.1.

(i) implies (iii). Indeed for all $x : X$ there exists $i : I$ such that $x \in U_i$ and then $T_x(X) = T_x(U_i)$ and $T_x(U_i) = 0$ by lemma 3.0.1.

(iii) implies (ii). Assume $x, y : X$, then:

$$x =_X y \leftrightarrow \Sigma_{y \in U_i} x =_{U_i} y$$

By lemma 3.0.1 we have that identity types in U_i is decidable, so $x =_X y$ is open. □

Now we generalise this to maps between schemes.

Proposition 3.0.3 A map between schemes is unramified if and only if its differentials are injective.

Proof The map df_x is injective if and only if its kernel is 0. By lemma 2.2.7, this means that df_x is injective for all $x : X$ if and only if:

$$\prod_{x:X} T_{(x,\mathrm{refl}_{f(x)})}(\mathrm{fib}_f(f(x))) = 0$$

On the other hand having fibers with trivial tangent space is equivalent to:

$$\prod_{y:Y} \prod_{x:X} \prod_{p:f(x)=y} T_{(x,p)}(\mathrm{fib}_f(y)) = 0$$

Both are equivalent by path elimination on p. □

4 Formally smooth and étale schemes

4.1 Smooth and étale maps between schemes

Note that it is immediate from the definition of smoothness that smooth maps induce surjections on tangent spaces. We have a converse when the domain is smooth.

Corollary 4.1.1 Let $f : X \to Y$ be a map between schemes with X smooth. Then the following are equivalent:

(i) The map f is smooth.

(ii) For all $x : X$, the induced map:
$$df : T_x(X) \to T_{f(x)}(Y)$$
is surjective.

Proof (i) implies (ii). Assume given a map $v : \mathbb{D}(1) \to Y$ such that $v(0) = f(x)$, then for all $t : \mathbb{D}(1)$ we have a map:
$$t = 0 \to \mathrm{fib}_f(v(t))$$
so since the f is smooth we merely have $w_t : \mathrm{fib}_f(v(t))$ such that $t = 0$ implies $w_t = 0$. We conclude using choice over $\mathbb{D}(1)$.

(ii) implies (i). Assume given $y : Y$ and $\epsilon : R$ such that $\epsilon^2 = 0$ and try to merely find a dotted lift in:

$$\begin{tikzcd}
\epsilon = 0 \arrow[r, "\phi"] \arrow[d] & \mathrm{fib}_f(y) \\
1 \arrow[ur, dashed] &
\end{tikzcd}$$

Since X is formally smooth we merely have an $x : X$ such that:
$$\prod_{p:\epsilon=0} \phi(p) = x$$
and therefore:
$$\epsilon = 0 \to y = f(x)$$
which means that $y \in N_1(f(x))$. We use lemma 2.4.3 to get that the map $N_1(x) \to N_1(f(x))$ merely has a section s sending $f(x)$ to x.

Then $s(y) : \mathrm{fib}_f(y)$ such that for all $p : \epsilon = 0$ we have that:
$$\phi(p) = x = s(f(x)) = s(y)$$
□

Corollary 4.1.2 Let $f : X \to Y$ be a map between schemes. Assume X is smooth. Then the following are equivalent:

(i) The map f is étale.

(ii) For all $x : X$, the induced map:
$$df : T_x(X) \to T_{f(x)}(Y)$$
is an iso.

Proof We apply proposition 3.0.3 and corollary 4.1.1. □

4.2 Smooth schemes have free tangent spaces

Lemma 4.2.1 Assume X is a smooth scheme. Then for any $x : X$ the type $T_x(X)$ is formally smooth.

Proof Consider $T(X) = X^{\mathbb{D}(1)}$ the total tangent bundle of X. We have to prove that the map:
$$p : T(X) \to X$$
is formally smooth. Both source and target are schemes, and the source is formally smooth because X is smooth and $\mathbb{D}(1)$ has choice. So by corollary 4.1.1 it is enough to prove that for all $x : X$ and $v : T_x(X)$ the induced map:
$$dp : T_{(x,v)}(T(X)) \to T_x(X)$$
is surjective.

Consider $w : T_x(X)$. By unpacking the definition of tangent spaces, we see that merely finding $w : T_{(x,v)}(T(X))$ such that $dp(w) = w$ means merely finding:
$$\phi : \mathbb{D}(1) \times \mathbb{D}(1) \to X$$

such that for all $t : \mathbb{D}(1)$ we have that:

$$\phi(0,t) = v(t)$$
$$\phi(t,0) = w(t)$$

But we know that there exists a unique:

$$\psi_{v,w} : \mathbb{D}(2) \to X$$

such that:

$$\psi_{v,w}(0,t) = v(t)$$
$$\psi_{w,v}(t,0) = w(t)$$

as defined in lemma 2.2.4.

Then the fact that X is smooth and that the fibers of:

$$\mathbb{D}(2) \to \mathbb{D}(1) \times \mathbb{D}(1)$$

are closed dense with $\mathbb{D}(1)\times\mathbb{D}(1)$ having choice means that there merely exists a lift of $\psi_{v,w}$ to $\mathbb{D}(1)\times\mathbb{D}(1)$, which gives us the ϕ we wanted. □

Lemma 4.2.2 Assume given a linear map:

$$M : R^m \to R^n$$

which has a formally smooth kernel. Then we can decide whether $M = 0$.

Proof Since $M = 0$ is closed, it is enough to prove that it is $\neg\neg$-stable to conclude that it is decidable by lemma 1.3.1 and lemma 1.3.2. Assume $\neg\neg(M = 0)$, then for any $x : R^m$ we have a dotted lift in:

$$\begin{tikzcd} M = 0 \arrow[r, "\mapsto x"] \arrow[d] & K \\ 1 \arrow[ur, dotted] & \end{tikzcd}$$

because K is formally smooth, so that we merely have $y : K$ such that:

$$M = 0 \to x = y$$

which implies that $\neg\neg(x = y)$ since we assumed $\neg\neg(M = 0)$.

Then considering a basis $(x_1, \cdots, x_n)$ of R^m, we get $(y_1, \cdots, y_n)$ such that for all i we have that $M(y_i) = 0$ and $\neg\neg(y_i = x_i)$. But then we have that $(y_1, \cdots, y_n)$ is infinitesimally close to a basis and that being a basis is an open proposition, so that $(y_1, \cdots, y_n)$ is a basis and $M = 0$. □

Lemma 4.2.3 Assume that K is a finitely copresented module that is also formally smooth. Then it is finite free.

Proof Assume a finite copresentation:

$$0 \to K \to R^m \overset{M}{\to} R^n$$

We proceed by induction on m. By lemma 4.2.2 we can decide whether $M = 0$ or not.

- If $M = 0$ then $K = R^m$ and we can conclude.
- If $M \neq 0$ then we can find a non-zero coefficient in the matrix corresponding to M, and so up to base change it is of the form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \widetilde{M} & \\ 0 & & & \end{pmatrix}$$

  But then we know that the kernel of M is equivalent to the kernel of $\widetilde{M}$, and by applying the induction hypothesis we can conclude that it is finite free. □

Proposition 4.2.4 Let X be a smooth scheme. Then for any $x : X$ we have that $T_x(X)$ is finite free.

Proof By lemma 4.2.1 we have that $T_x(X)$ is formally smooth, so that we can conclude by lemma 4.2.3.□

The dimension of $T_x(X)$ is called the dimension of X at x. By boundedness any smooth scheme is a finite sum of smooth scheme of a fixed dimension.

4.3 Standard étale and standard smooth schemes

Definition 4.3.1 A standard smooth scheme of dimension k is an affine scheme of the form:

$$\mathrm{Spec}\left((R[X_1,\cdots,X_n,Y_1,\cdots Y_k]/P_1,\cdots,P_n)_G\right)$$

where G divides the determinant of:

$$\left(\frac{\partial P_i}{\partial X_j}\right)_{1\leq i,j\leq n}$$

in:

$$R[X_1,\cdots,X_n,Y_1,\cdots Y_k]/P_1,\cdots,P_n$$

Definition 4.3.2 A standard smooth scheme of dimension 0 is called a standard étale scheme.

Lemma 4.3.3 Standard étale schemes are étale.

Proof Assume given a standard étale algebra:

$$(R[X_1,\cdots,X_n]/P_1,\cdots,P_n)_G$$

and write:

$$P : R^n \to R^m$$

for the map induced by $P_1,\cdots,P_m$.

Assume given $\epsilon : R$ such that $\epsilon^2 = 0$, we need to prove that there is a unique dotted lifting in:

$$\begin{tikzcd}
R/\epsilon & (R[X_1,\cdots,X_n]/P_1,\cdots,P_n)_G \arrow[l, "x"'] \arrow[dl, dotted] \\
R \arrow[u] &
\end{tikzcd}$$

This means that for all $x : R^n$ such that $P(x) = 0 \bmod \epsilon$ and $G(x)$ invertible modulo ϵ (or equivalently $G(x)$ invertible), there exists a unique $y : R^n$ such that:

- We have $x = y \bmod \epsilon$.
- We have $P(y) = 0$.
- We have $G(y) \neq 0$ (this is implied by $x = y \bmod \epsilon$ and $G(x) \neq 0$).

First we prove existence. For any $b : R^n$ we compute:

$$P(x + \epsilon b) = P(x) + \epsilon\ dP_x(b)$$

We have that $P(x) = 0 \bmod \epsilon$, say $P(x) = \epsilon a$. Then since $G(x) \neq 0$ and $\det(dP)$ divides G, we have that dP_x is invertible. Then taking $b = -(dP_x)^{-1}(a)$ gives a lift $y = x + \epsilon b$ such that $P(y) = 0$.

Now we check unicity. Assume y, y' two such lifts, then $y = y' \bmod \epsilon$ and we have:

$$P(y) = P(y') + dP_{y'}(y - y')$$

and $P(y) = 0$ and $P(y') = 0$ so that:

$$dP_{y'}(y - y') = 0$$

But $G(y') \neq 0$ so $dP_{y'}$ is invertible and we can conclude that $y = y'$. □

Lemma 4.3.4 Any standard smooth scheme of dimension k is formally smooth of dimension k.

Proof The fibers of the map:

$$\mathrm{Spec}\left((R[X_1,\cdots,X_n,Y_1,\cdots Y_k]/P_1,\cdots,P_n)_G\right) \to \mathrm{Spec}(R[Y_1,\cdots Y_k])$$

are standard étale, so the map is étale by lemma 4.3.3. Since:

$$\mathrm{Spec}(R[Y_1,\cdots Y_k]) = \mathbb{A}^k$$

is smooth by **??**, we can conclude it is smooth using **??**.

For the dimension we use lemma 2.2.3 and corollary 4.1.2. □

4.4 Smooth schemes are locally standard smooth

Proposition 4.4.1 A scheme is smooth of dimension k if and only if it has a finite open cover by standard smooth schemes of dimension k.

Proof We can assume the scheme X affine, say of the form:

$$X = \mathrm{Spec}(R[X_1, \cdots, X_m]/P_1, \cdots, P_l)$$

By proposition 4.2.4, for any $x : X$ we have that dP_x has free kernel. We partition by the dimension k of the kernel. Then by **??** we know that dP_x has rank $n = m - k$ for every x.

We cover X according to which n-minor is invertible, so that up to a rearranging of variables and polynomials we can assume that:

$$X = \mathrm{Spec}(R[X_1, \cdots, X_n, Y_1, \cdots, Y_k]/P_1, \cdots, P_n, Q_1, \cdots, Q_l)$$

where we have:

$$dP_{x,y} = \begin{pmatrix} \left(\frac{\partial P}{\partial X}\right)_{x,y} & \left(\frac{\partial P}{\partial Y}\right)_{x,y} \\ \left(\frac{\partial Q}{\partial X}\right)_{x,y} & \left(\frac{\partial Q}{\partial Y}\right)_{x,y} \end{pmatrix}$$

where we used the notation:

$$\left(\frac{\partial P}{\partial X}\right)_{x,y} = \left(\left(\frac{\partial P_i}{\partial X_j}\right)_{x,y}\right)_{i,j}$$

so that $\frac{\partial P}{\partial X}$ is invertible of size n. Moreover by lemma 2.5.2 we get:

$$\left(\frac{\partial Q}{\partial Y}\right)_{x,y} = \left(\frac{\partial Q}{\partial X}\right)_{x,y} \left(\frac{\partial P}{\partial X}\right)^{-1}_{x,y} \left(\frac{\partial P}{\partial Y}\right)_{x,y}$$

which will be useful later.

Now we prove that for any $(x, y) : R^{n+k}$ such that $P(x, y) = 0$ it is decidable whether

$$Q(x, y) = 0$$

To do this it is enough to prove that:

$$(Q_1(x, y), \cdots, Q_l(x, y))^2 = 0 \to (Q_1(x, y), \cdots, Q_l(x, y)) = 0$$

Assuming $(Q_1(x, y), \cdots, Q_l(x, y))^2 = 0$, by smoothness there is a dotted lifting in:

$$\begin{tikzcd}
R/(Q_1(x,y), \cdots, Q_l(x,y)) & \mathrm{Spec}(R[X_1, \cdots, X_n, Y_1, \cdots, Y_k]/P_1, \cdots, P_n, Q_1, \cdots, Q_l) \arrow[l, "{(x,y)}"'] \arrow[dl, dotted, "{(x',y')}"] \\
R \arrow[u] &
\end{tikzcd}$$

Let us prove that $Q(x, y) = 0$. Indeed we have $(x, y) \sim_1 (x', y')$ so that we have:

$$P(x, y) = P(x', y') + \left(\frac{\partial P}{\partial X}\right)_{x',y'} (x - x') + \left(\frac{\partial P}{\partial Y}\right)_{x',y'} (y - y')$$

$$Q(x, y) = Q(x', y') + \left(\frac{\partial Q}{\partial X}\right)_{x',y'} (x - x') + \left(\frac{\partial Q}{\partial Y}\right)_{x',y'} (y - y')$$

Then we have $P(x, y) = 0$, $P(x', y') = 0$ and $Q(x', y') = 0$. From the first equality we get:

$$x - x' = -\left(\frac{\partial P}{\partial X}\right)^{-1}_{x',y'} \left(\frac{\partial P}{\partial Y}\right)_{x',y'} (y - y')$$

so that from the second we get:

$$Q(x, y) = -\left(\frac{\partial Q}{\partial X}\right)_{x',y'} \left(\frac{\partial P}{\partial X}\right)^{-1}_{x',y'} \left(\frac{\partial P}{\partial Y}\right)_{x',y'} (y - y') + \left(\frac{\partial Q}{\partial Y}\right)_{x',y'} (y - y')$$

so that $Q(x,y) = 0$ as we have seen previously that:

$$\left(\frac{\partial Q}{\partial Y}\right)_{x',y'} = \left(\frac{\partial Q}{\partial X}\right)_{x',y'} \left(\frac{\partial P}{\partial X}\right)^{-1}_{x',y'} \left(\frac{\partial P}{\partial Y}\right)_{x',y'}$$

From the decidability of $Q(x,y) = 0$ we get that X is an open in:

$$\mathrm{Spec}(R[X_1, \cdots, X_n, Y_1, \cdots, Y_k]/P_1, \cdots, P_n)$$

so it is of the form $D(G_1, \cdots, G_n)$, and we have an open cover of our scheme by pieces of the form:

$$\mathrm{Spec}((R[X_1, \cdots, X_n, Y_1, \cdots, Y_k]/P_1, \cdots, P_n)_G)$$

Where $P_i(x) = 0$ for all i and $G(x) \neq 0$ implies:

$$\det(\mathrm{Jac}(P_1, \cdots, P_n)_x) \neq 0$$

We write:

$$F(x) = \det(\mathrm{Jac}(P_1, \cdots, P_n)_x)$$

Then for all $x : \mathrm{Spec}(R[X_1, \cdots, X_n, Y_1, \cdots, Y_k]/P_1, \cdots, P_n)$ we have that:

$$(G(x) \neq 0) \to (F(x) \neq 0)$$

so that there exists n such that:

$$F(x) | G(x)^n$$

and using boundedness we get N such that for all $x : \mathrm{Spec}(R[X_1, \cdots, X_n, Y_1, \cdots, Y_k]/P_1, \cdots, P_n)$ we have:

$$F(x) | G(x)^N$$

and we conclude that F divides G^N in $R[X_1, \cdots, X_n, Y_1, \cdots, Y_k]/P_1, \cdots, P_n$. So by replacing G by G^N, we get standard smooth pieces. $\square$

Corollary 4.4.2 A scheme is formally étale if and only if it has a cover by standard étale schemes.

Proof By corollary 3.0.2 we know that a scheme is formally étale if and only if it is smooth of dimension 0. Then we just apply proposition 4.4.1. $\square$

Index

References

[Chr+20] J. Daniel Christensen et al. "Localization in Homotopy Type Theory". In: (2020). arXiv: 1807.04155 [math.AT]. URL: https://arxiv.org/abs/1807.04155 (cit. on p. 3).

[LQ15] Henri Lombardi and Claude Quitté. *Commutative Algebra: Constructive Methods*. Springer Netherlands, 2015. DOI: 10.1007/978-94-017-9944-7. URL: https://arxiv.org/abs/1605.04832 (cit. on p. 3).

[RSS20] Egbert Rijke, Michael Shulman, and Bas Spitters. "Modalities in homotopy type theory". In: *Logical Methods in Computer Science* Volume 16, Issue 1 (Jan. 2020). DOI: 10.23638/LMCS-16(1:2)2020. URL: https://lmcs.episciences.org/6015 (cit. on p. 2).