Synthetic Stone Duality

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Abstract

In synthetic algebraic geometry (SAG) [CCH23], we study finitely presented algebras over a commutative ring. In this work, we study countably presented Boolean algebras instead. Where the finitely presented algebras over a commutative ring induce a Zariski topos, the countably presented Boolean algebras induce the topos of light condensed sets [CS24]. [CCH23] proposes an axiomatization of the Zariski topos in univalent homotopy type theory [Pro13]. In this work, we propose similar axioms, which we expect to be modelled by light condensed sets.

(The following is a collection of notes on work in progress.)

Introduction

Definition 0.1 A countably presented Boolean algebra B is a Boolean algebra such that there merely are countable sets I, J, a set of generators $g_i,\ i \in I$ and a set $f_j,\ j \in J$ of Boolean expressions over these generators such that B is equivalent to the quotient of the free Boolean algebra over the generators by the relations $f_j = 0$.

If I, J are finite, we call B a finitely presented Boolean algebra.

Remark 0.2 As Boolean algebras are rings, any relation of the form $f = g$ with both f, g Boolean expressions can be written as $h = 0$ with $h = f - g$ a Boolean expression.

We can express a countably presented Boolean algebra as the colimit of a finitely presented Boolean algebra. This is the formulation closer to [CS24].

Lemma 0.3 B is a countably presented Boolean algebra iff it merely is the colimit of a sequence of finitely presented Boolean algebras.

Proof First, assume a sequence of finitely presented Boolean algebras. We need to show that the colimit is a countably presented Boolean algebra.

- The set of generators for the colimit is the colimit of the sets of generators.
- The set of relations for the colimit is the union of the sets of relations. After all, any expression f that becomes 0 somewhere in the sequence will will be coprojected to 0 in the colimit. And as any equality that holds in the colimit uses finitely many elements, it must already hold somewhere in the sequence.

Note that both colimits over countably many finite sets are countable. Hence the colimit is countably represented.

Conversely, given a countably presented Boolean algebra B, we need to give a sequence and show it's colimit is B. For our sequence, we assume we have an enumeration of the generators of B. We let G_n be given by the first n generators. Let R_n be the relations involving these generators, of which there are only finitely many. We define $B_n = G_n/R_n$, which is a finitely presented Boolean algebra. The embedding of the first n generators into the first m generators gives us a map $B_n \to B_m$ whenever $n \leq m$. Because these morphisms are compatible, this defines a sequence of Boolean algebras. We claim the colimit of this sequence is B.

Any element in B can be expressed as Boolean combination of finitely many generators, which must occur in some B_n, and thus in the colimit. Whenever the images of two elements in the colimit are equal, they are already equal in some B_m, hence it follows from a finite subset of the relations for B that the elements are equal, hence the elements are equal in B. Thus we have an embedding from B into the colimit.

Any element in the colimit already appears in some B_n, and hence is a finite expression using generators from B, thus occurs in B is as well. Suppose two elements in the colimit correspond in this manner to the same element in B. Then their equality follows from the relations of B. By compactness in the meta-theory, their equality must follow from a finite subset of the relations from B, hence there is some B_m where both elements are equal, and they are equal in the colimit as well. Thus the colimit embeds into B.

We conclude that B and the colimit are isomorphic Boolean algebras. □

Definition 0.4 We call an object K (countably) compact if for every sequence $A = colim A_n$, we have $A^K = colim A_n^K$.

Lemma 0.5 Finitely presented algebras are compact in the category of algebras.

The following uses Dependent Choice.

Lemma 0.6 If $A \to B$ is injective between countably presented Boolean algebras, we can write it as colimit of injections between finitely presented Boolean algebras.

In SAG, we deal with a fixed commutative ring R. For this project, the role of R is taken over by the Boolean algebra $2 = 1 + 1$. Note that we don't need to postulate an alternative for the **Loc** axiom. We write Boole the type of countably presented Boolean algebras. Note that as each Boolean algebra is a Set, we Boole is a subtype of $hSet$. Also, as being countable is a notion independent of universes, Boole is independent of universes. Finally, note that Boole has a natural category structure.

Definition 0.7 For B a countably presented Boolean algebra, we define $Sp(B)$ as the set of Boolean morphisms from B to 2.

An example of an element of Boole is the free algebra C on countably many generators. The corresponding set $Sp(C)$ is then Cantor space $2^{\mathbb{N}}$.

Another example is the algebra B_∞ generated by p_n with relations $p_n p_m = 0$ for $n \neq m$. The corresponding set $Sp(B_\infty)$ is the set $\mathbb{N}_\infty$ of binary sequences with at most one element $\neq 0$.

Axiom 1 (Stone duality)
For any countably presented Boolean algebra B, the evaluation map $B \to 2^{Sp(B)}$ is an isomorphism.

Definition 0.8 We define the predicate on types $\mathsf{isStone}$ by

$$\mathsf{isStone}(X) = \sum_{B:Boole} X = Sp(B) \tag{1}$$

A type X is called *Stone* if $\mathsf{isStone}(X)$ is inhabited.

Stone types will take over the role of affine scheme from [CCH23], and we repeat some results here. Analogously to Lemma 3.1.2, for X Stone, we have $X = Sp(2^X)$. Proposition 2.2.1 now says that Sp gives an equivalence

$$Hom_{\mathsf{Boole}}(A, B) = (Sp(B) \to Sp(A)) \tag{2}$$

By [**HoTT**; **p TODO**], it follows that Sp is an embedding from Boole to any universe of types. Its image, Stone also has a natural category structure. The map Sp defines then an anti-equivalence of categories between Boole and Stone.

Any Stone set has a natural topology, where basic open are decidable subsets.

Proposition 0.9 Any map $f : Sp(B) \to \mathbb{N}$ is uniformely continuous.

Proof For each natural number n, the fiber $f^{-1}(n)$ is a decidable subset of $Sp(B)$. Via the isomorphism $B \to 2^{Sp(B)}$, this corresponds to an element e_n of B. We have $e_n e_m = 0$. Furthermore the quotient B' of B by the relations $e_n = 0$ is such that $Sp(B') = 0$ and hence $1 = 0$ in B', so we have N such that $1 = \vee_{i<N} e_i$. □

In formal/point-free topology, we consider that a Boolean algebra B represents a Stone space $Sp(B)$ and a map $Sp(B') \to Sp(B)$ is represented by a map $B \to B'$; the map $Sp(B') \to Sp(B)$ is then said to be *formally surjective* if the corresponding map $B \to B'$ is injective. In the topos of light condensed sets, this becomes a true duality.

Proposition 0.10 Markov's Principle holds, if we have $\neg\forall_n\alpha(n) = 0$ then we have $\exists_n\alpha(n) = 1$.

Proof Let B be the Boolean algebra presented by $\alpha(n)$. We have $Sp(B) = \emptyset$ and hence by duality B is trivial, which means that we have n such that $\alpha(n) = 1$. □

Axiom 2 (Surjections are Formal Surjections)
A map $f : Sp(B') \to Sp(B)$ is surjective iff the corresponding map $B \to B'$ is injective.

Another way to state this axiom is that epimorphisms in the category Stone are exactly the surjective maps.

Yet another formulation is $(\neg\neg X) \to \|X\|$ for X Stone space. If we think of an algebra in Boole as a proposition theory, this expresses a form of *completness*: any non inconsistent theory has a model.

An example of a surjective map (since it is an epimorphism, since it corresponds to a monomorphism via the anti-equivalence between Stone and Boole) is the map sum of the maps $\mathbb{N}_\infty \to \mathbb{N}_\infty$ sending n to $2n$ (resp. n to $2n + 1$). This map has no section. This shows that $\mathbb{N}_\infty$ is not projective.

Here is another way to formulate this result.

Proposition 0.11 LLPO is a consequence of Axioms 1 and 2.

Conversely, *with* Dependent Choice, LLPO implies Axiom 2, since it implies completeness of propositional logic.

A consequence of this characterisation of surjective maps is the following.

Proposition 0.12 The image of any map between two Stone types is Stone.

Here is an example showing how to use this axiom. A closed subset of a Stone set is given by a countable intersection of decidable subset.

Proposition 0.13 Let $f : X' \to X$ a surjective map and F_n a decreasing sequence of closed subsets of X' such that each restriction $f_{|F_n}$ is surjective. Then if $F = \cap_n F_n$ the restriction $f_{|F}$ is still surjective.

Proof Dually, we have an injective map $i : B \to B'$ with an increasing sequence I_n of ideals of B' such that $b = 0$ if $i(b) = 0$ mod. I_n. The subset F corresponds to the ideal $I = \cup_n I_n$. If $i(b) = 0$ mod. I then we have $i(b) = 0$ mod. I_n for some n and $b = 0$. □

Axiom 3 (Local choice)
Whenever X Stone and $E \twoheadrightarrow X$ surjective, then there is some Y Stone, a surjection $Y \twoheadrightarrow X$ and a map $Y \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[d, two heads] & \\ X & Y \arrow[l, two heads] \arrow[ul] \end{tikzcd} \tag{3}$$

The last axiom is Dependent Choice.

Axiom 4 (Dependent Choice)
Given a family of types E_n and $R_n : E_n \to E_{n+1} \to \mathcal{U}$ such that for all n and $x : E_n$ there exists $y : E_{n+1}$ with $p : R_n\ x\ y$ then given $x_0 : E_0$ there exists $u : \Pi_{n:\mathbb{N}} E_n$ and $v : \Pi_{n:\mathbb{N}} R_n\ (u\ n)\ (u\ (n+1))$ and $u\ 0 = x_0$.

One basic result about the category Boole, the existence of retraction for non empty closed subset inclusion holds only *non* constructively and in our setting we can prove the following.

Proposition 0.14 It is not the case that for all closed proposition p the inclusion $1 + p \to 1 + 1$ has a retraction.

Proof This implies that all closed propositions are decidable and the proposition $x = \infty$ for x in $\mathbb{N}_\infty$ is a closed proposition which is not decidable. □

We can define the set Closed of closed propositions, where a proposition is closed iff it is equivalent to the proposition $\forall_n\alpha(n) = 0$ for some α in $2^{\mathbb{N}}$.

Theorem 0.15
Monomorphisms in Stone are classified by Closed.

We have seen that $\mathbb{N}_\infty$ is not projective. Using Local and Dependent Choice, David noticed that Scholze's argument about $\mathbb{Z}[\mathbb{N}_\infty]$ cannot be made internal.

Theorem 0.16
$\mathbb{Z}[\mathbb{N}_\infty]$ is *not* projective in the category of Abelian Groups.

1 Omniscience principles

Lemma 1.1 For (A_n) a family of decidable subsets, we have $(\bigcup_{n:\mathbb{N}} A_n)^C = \bigcap_{n:\mathbb{N}}(A_n^C)$ and $\bigcup_{n:\mathbb{N}}(A_n^C) = (\bigcap_{n:\mathbb{N}} A_n)^C$

Proof

- Let $x \notin \bigcup_{n:\mathbb{N}} A_n$. Then for every $n : \mathbb{N}$, we cannot have $x \in A_n$ and thus $x \in A_n^C$ by decidability of A_n. Thus $x \in \bigcap_{n:\mathbb{N}}(A_n^C)$. Therefore
$$(\bigcup_{n:\mathbb{N}} A_n)^C \subseteq \bigcap_{n:\mathbb{N}}(A_n^C).$$
- Suppose that for every $n : \mathbb{N}$, we have $x \notin A_n$. There does not exist an $n : \mathbb{N}$ with $x \in A_n$. Thus
$$\bigcap_{n:\mathbb{N}}(A_n^C) \subseteq (\bigcup_{n:\mathbb{N}} A_n)^C$$
- Suppose there exists some n with $x \in A_n^C$. Then it cannot be the case that $x \in A_m$ for all $m : \mathbb{N}$. Thus
$$\bigcup_{n:\mathbb{N}}(A_n^C) \subseteq (\bigcap_{n:\mathbb{N}} A_n)^C$$
- Suppose that $x \in (\bigcap_{n:\mathbb{N}} A_n)^C$. Then define the binary sequence α by $\alpha(i) = 1$ iff i is the first index such that $x \notin A_i$. This is well-defined as A_n is decidable for all $n : \mathbb{N}$. If $\alpha(i) = 0$ for all i, then $x \in A_i$ for all i. Thus under our assumption $x \in (\bigcap_{n:\mathbb{N}} A_n)^C$, we cannot have that $\alpha(i) = 0$ always. By Markov, there then exists an i such that $\alpha(i) = 1$. Thus $x \notin A_i$ for some i. We conclude that.
$$(\bigcap_{n:\mathbb{N}} A_n)^C \subseteq \bigcup_{n:\mathbb{N}}(A_n^C)$$

Note that we only needed decidability for the first and last bullet point, and only the last bullet point used countability (and of course Markov's principle).

2 Topology

Definition 2.1 The image of a map $f : X \to Y$ between types is given by
$$\mathrm{im}(f) :\equiv \sum_{y:Y} \exists_{x:X} f(x) = y$$
and yields a factorization using the canonical maps:
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & \mathrm{im}(f) \arrow[ur] & \end{tikzcd}$$

Proposition 2.2 The image $\mathrm{im}(f)$ of a map $f : X \to Y$ between stone spaces $X = \mathrm{Spec}(B), Y = \mathrm{Spec}(B')$ is a subtype of the form $\mathrm{Spec}(B'/I) \subseteq \mathrm{Spec}(B')$ for a countably generated ideal $I \subseteq B'$.

Proof TODO □

Definition 2.3

(a) A proposition P is *closed* if there merely is a sequence $s : 2^{\mathbb{N}}$ such that P is equivalent to $s = 0$.

(b) Let X be an arbitrary type. A subtype $C \subseteq X$ is *closed* if $C(x)$ is a closed proposition for all $x : X$.

Proposition 2.4 Let X be a stone space and $C \subseteq X$ a subset. Then the following are equivalent:

(i) C is closed.

(ii) C is a countable intersection of decidable subsets.

(iii) There is a countable family of functions $(f_i : X \to 2)_{i:\mathbb{N}}$ such that
$$C = \{x : X \mid \forall i. f_i(x) = 0\}.$$

Proof TODO using **??** and Proposition 2.2. □

2.1 Compact Hausdorff

Definition 2.5 Let S be Stone, $C \subseteq S$. Then C is open if it is the countable union of decidable subsets.

Lemma 2.6 For S Stone and $C \subseteq S$, C is closed iff it's complement is open and C is open iff it's complement is closed.

Proof This follows from the fact that the complement of a decidable subset is decidable and Lemma 1.1.□

Lemma 2.7 For S Stone, any cover by opens merely has a finite subcover.

Proof Let $S = \bigcup_{i:I} A_i$ be a cover of S by open sets. Assume furthermore $S = Sp(B)$. As every open is the union of decidable subsets, we may assume A_i decidable, and thus corresponding to points $a_i \in B$. These points are such that $1 = \bigvee_{i:I} a_i$. As B is countably presented, it is countable. Thus $(a_i)_{i:I}$ is a countable set. The morphism $I \to B$ is surjective, and as we're proving a proposition, we may use type-theoretic AC to give a countable subset $I_0 \subseteq I$ such that $\bigvee_{i:I_0} a_i = 1$ as well. So $S = \bigcup_{i:I_0} A_i$ for I_0 countable. □

Note that the basic clopens are not the only clopens. I.e. not every set that is both a countable intersection of decidable subsets and a countable union of decidable subset is itself decidable. In B_∞, we can describe the even numbers both as the infinite meet of cofinite sets excluding odd numbers up to n and the join of finite sets including even numbers up to n. But the even numbers do not themselves for an element of B_∞. Thus the set of maps $B \to 2$ sending every χ_{2n} to 1 is clopen but not decidable.

Definition 2.8 We define a type X to be compact Hausdorff iff X is the quotient of a stone space S by a closed equivalence relation. A subtype $C \subseteq X$ is closed respectively open iff it's pre-image under the quotient map is.

Lemma 2.9 In a compact Hausdorff, closed sets are closed under intersection.

Lemma 2.10 In a Compact Hausdorff space, the complement of an open is closed, and the complement of a closed is open.

Proof Let $e : S \to X$ be the quotient map of a Stone space by a closed equivalence relation. and let $(A_n)_{n:\mathbb{N}}$ be a countable family of decidable subsets in S.

First, we claim that $X - \bigcup_{n:\mathbb{N}} e(A_n)$ is closed in X. □

Lemma 2.11 Whenever X is compact Hausdorff, F_0, F_1 are closed and disjoint, there exist G_0, G_1 disjoint clopen such that $F_i \subseteq X - G_{1-i}$ and $G_0 \cup G_1 = X$.

2.2 Intersection of closed in compact Hausdorff

Lemma 2.12 In a compact Hausdorff, closed sets are closed under intersection.

Proof

Lemma 2.13 For S Stone, $D \subseteq S$ decidable, $\sim$ a closed equivalence relation on S, the set $\{x : S | \exists y : D(x \sim y)\}$ is closed.

Proof

Lemma 2.14 For S Stone, $D \subseteq S$ decidable, $\sim$ a decidable equivalence relation on S, the set $\{x : S | \exists y : D(x \sim y)\}$ is closed.

Proof Let $B = 2^S$, so $S = Sp(B)$. As D is decidable, there is some $n : \mathbb{N}$ such that $D(y)$ only depends on $y|_n$.

As $\sim$ is decidable, there is a finite set $I_0 \subseteq \mathbb{N}$, such that $x \sim y = \prod_{i:I_0} x(i) = y(i)$.

Thus

$$\exists (y : D)(x \sim y) = ||\Sigma(y : 2^{\mathbb{N}}) y(b) = 1 \wedge \prod (i : I_0) x(i) = y(i)||$$

Lemma 2.15 Let S Stone, then $D \subseteq S$ is closed iff $D \subseteq S \subseteq 2^{\mathbb{N}}$ is closed.

Proof Follows immediately from countable intersection of basic clopen. □

3 Analysis

3.1 Convergence

Topological convergence In this section, X is a Stone space.

Definition 3.1 A sequence in X is a map $\mathbb{N} \to X$.

Definition 3.2 Let α be a sequence in X. We say that x is the limit of α iff for any open $U \subseteq X$ containing x, there merely is an $N : \mathbb{N}$ such that for $n \geq N$, we have $x_n \in U$.

Closed spaces contain their limits

Lemma 3.3 Let $x : 2^{\mathbb{N}}$ and $D \subseteq 2^{\mathbb{N}}$ be a decidable subset. Suppose that for each open $U \subseteq X$ with $U(x)$, we merely have some $y_U \in D \cap U$. Then $x \in D$.

Proof Because D is a subtype, $x \in D$ is a proposition, and we will use existence whenever we have mere existence. Because D is decidable, there merely exists an $n : \mathbb{N}$ such that whenever $x =_n y$, we have $D(x) \leftrightarrow D(y)$. Consider the open U_n given by $x =_n \cdot$. By assumption, there merely is some $y \in D \cap U_n$. so $D(y)$ and $x =_n y$, hence $D(x)$. □

Corollary 3.4 Let $\iota : D \hookrightarrow 2^{\mathbb{N}}$ be the inclusion map of a decidable subset, let α be a sequence in D, and suppose that $\alpha \circ \iota$ has a limit x in $2^{\mathbb{N}}$. Then $x \in D$.

Corollary 3.5 Using (ii) from Proposition 2.4 it follows that any closed subset of a Cantor space contains all of it's limit points.

Remark 3.6 The converse is not true. It is not the case that if a subset of a Stone space contains its limits, it is necessarily closed. For any propostion p, we have the subset of Cantor space given by $A = \{x : 2^{\mathbb{N}} | p\}$. If A was closed, p would be equivalent to a proposition of the form $\alpha = 0$. However, not all propositions are of this form. So A needn't be closed. But if a sequence in A exists and has a limit, because the sequence exists, p must hold and thus the limit is contained in A also.

Extensional convergence

Definition 3.7 Let B_∞ be the Boolean algebra on countably many generators $(p_n)_{n:\mathbb{N}}$ over the equivalence $p_n \wedge p_m = 0$ whenever $n \neq m$.

Definition 3.8 We denote $\mathbb{N}_\infty$ be the spectrum of B_∞.

Lemma 3.9 B_∞ is isomorphic with the Boolean algebra of finite/cofinite subsets of $\mathbb{N}$.

Proof To go from B_∞ to subsets of $\mathbb{N}$, we send the generators p_n to the singleton $\{n\}$, which are clearly finite. We call the induced Boolean operation f.

To go from finite/cofinite subsets of $\mathbb{N}$ to B_∞, a finite subset I of $\mathbb{N}$ is sent to the element $\bigvee_{i \in I} p_i$, and a cofinite subset J is sent to the element $\bigwedge_{i \in J^C} \neg p_i$. We call this function g and we need to show that g is a Boolean morphism.

- By deMorgan's laws, g preserves $\neg$.
- To see that g respects $\vee$, we need to check three cases
  - If both I, J are finite, then
$$g(I \cup J) = \bigvee_{i \in I \cup J} p_i = \bigvee_{i \in I} p_i \vee \bigvee_{j \in J} p_j \tag{4}$$
  - If both I, J are cofinite, we have
$$g(I) \vee g(J) = \bigwedge_{i \in I^C} \neg p_i \vee \bigwedge_{j \in J^C} \neg p_j = \bigwedge_{i \in I^C} \bigwedge_{j \in J^C} (\neg p_i \vee \neg p_j) \tag{5}$$
    Now note that $\neg p_i \vee \neg p_j = \neg(p_i \wedge p_j)$, which is 1 if $i \neq j$ and p_i if $i = j$. We can leave 1 out of the meet, and we are left with the intersection of I^C and J^C, so
$$g(I) \vee g(J) = \bigwedge_{i \in (I^C \cap J^C)} \neg p_i = \bigwedge_{i \in (I \cup J)^C} \neg p_i \tag{6}$$
    as the union of I and J is also cofinite, this equals $g(I \cup J)$.

– If I is finite and J cofinite, we have

$$g(I) \vee g(J) = (\bigvee_{i \in I} p_i) \vee (\bigwedge_{j \in J^C} \neg p_j) = \bigwedge_{j \in J^C} (\bigvee_{i \in I} (p_i \vee \neg p_j)) \tag{7}$$

If $i \neq j$, then $p_i \wedge p_j = 0$, hence $\neg p_j \geq p_i$ and $p_i \vee \neg p_j = \neg p_j$ If $i = j$, then $p_i \vee \neg p_j = 1$.

- The case for $\wedge$ is completely dual to the case for $\vee$.

We conclude that g is a Boolean morphism. Furthermore, g and f are clearly inverses, thus the Boolean algebras are isomorphic. □

Lemma 3.10 Any element of B_∞ can be written as either $\bigvee_{i \in I} p_i$ or as $\bigwedge_{j \in J} \neg p_j$ for finite $I, J \subseteq \mathbb{N}$.

Proof Remark that whenever $n \neq m$, we have that $\neg p_n \geq p_m$ as $p_m \wedge p_n = 0$. □

There is canonical embedding $\mathbb{N} \hookrightarrow \mathbb{N}_\infty$, wich sends n to the unique function χ_n sending p_n to 1. We denote $\infty \in \mathbb{N}_\infty$ for the function which is constantly 0. By Proposition 0.10, if an element is not ∞, it comes from the embedding $\mathbb{N} \hookrightarrow \mathbb{N}_\infty$.

Lemma 3.11 Let U be an open subset of $\mathbb{N}_\infty$ containing ∞. Then there merely exists an $N : \mathbb{N}$ such that whenever $n \geq N$, $\chi_n \in U$ as well.

Proof It is sufficient to prove the lemma for U a basic open. Assume $b : B_\infty$ is such that $U = \{\phi : B_\infty \to 2 | \phi(b) = 1\}$. Assume furthermore that $\infty \in U$. by Lemma 3.10, b can have two forms. If $b = \vee_{i \in I} p_i$, then as $\infty(b) = 0$, we must have $I = \emptyset$, and thus $b = 0$, which means U is empty, contradicting $\infty \in U$. Therefore, b must be of the form $\wedge_{j \in J} \neg p_j$. Note that for $N = \max J + 1$, whenever $n > J$, χ_n sends b to 1. Thus $\chi_n \in U$ as well, and we are done. □

Definition 3.12 Let α be a sequence in X, we say that α is convergent iff there exists an extension.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\alpha"] \arrow[d, hook] & X \\ \mathbb{N}_\infty \arrow[ur, dashed] & \end{tikzcd} \tag{8}$$

Proposition 3.13 A sequence is convergent iff it has a limit

Proof Let α be convergent, with extension $\overline{\alpha}$. we claim that $\overline{\alpha}(\infty)$ is a limit of α. Let $U \subseteq X$ be an open containing x. As $\overline{\alpha}^{-1}(U)$ is an open subset of $\mathbb{N}_\infty$ containing ∞, Lemma 3.11 tells us there exists some N such that $[N, \infty] \subseteq \overline{\alpha}^{-1}(U)$. Thus there exists an N such that for $n \geq N$, we have $\alpha(n) \in U$, as required.

Conversely, suppose α has limit x. Assume $X = Sp(B)$, and let $b \in B$. Then b corresponds to a decidable subset $U \subseteq X$. For any decidable subset $U \subseteq X$, we have $\alpha^{-1}(U)$ a decidable subset of $\mathbb{N}$. We claim that $\alpha^{-1}(U)$ is either finite or cofinite. As U is decidable, we can decide wheter $x \in U$. If $x \in U$, $\alpha^{-1}(U)$ is cofinite, as $\alpha(n) \in U$ for all $n \geq N$ for some N. If $x \notin U$, we have $x \in U^C$, which is also decidable and therefore $\alpha^{-1}(U^C)$ is cofinite. As $\alpha^{-1}(U)^C = \alpha^{-1}(U^C)$, it follows that $\alpha^{-1}(U)$ is finite. Thus $\alpha^{-1}(U)$ is finite or cofinite for any decidable subset $U \subseteq X$. Finite and cofinite subsets of $\mathbb{N}$ correspond to elements of B_∞. Therefore, α induces a map $B \to B_\infty$, which corresponds to a map $\overline{\alpha} : \mathbb{N}_\infty \to X$.

We claim that $\overline{\alpha}$ extends α. Denote ι for the map $\mathbb{N} \to \mathbb{N}_\infty$. We need to show that $\overline{\alpha} \circ \iota = \alpha$. By definition, we have that $(\overline{\alpha} \circ \iota)^{-1}(U) = \alpha^{-1}(U)$ for any decidable $U \subseteq X$. □

Lemma 3.14 Whenever $S = Sp(B)$ Stone, $f, g : A \to S$, and $f^{-1}(U) = g^{-1}(U)$ for any decidable $U \subseteq S$, we have $f = g$.

Proof By our assumption, we have for all $a : A$ that $f(a) \in U \iff g(a) \in U$ for any decidable $U \subseteq X$. Such U correspond to $b : B$. and $f(a) \in U \iff f(a)(b) = 1$. So the functions $f(a), g(a) : B \to 2$ are such that $f(a)(b) = g(a)(b)$ for all $b : B$. This holds for all $a : A$ and by two uses of function extensionality we may conclude $f = g$. □

3.2 The interval

3.3 The Cauchy reals

The goal of this section is to introduce the real numbers in a constructive setting, following the definition given in [BB85] with some small adaptations. We will later use this definition to show that the interval $[0,1]$ is compact Hausdorff in the sense of **??**.

We will assume we are given natural and rational numbers, with decidable (in)equalities working as expected.

Definition 3.15 A **Cauchy sequence** is a sequence $x : \mathbb{N} \to \mathbb{Q}$ such that for any $n, m : \mathbb{N}$, we have $|x_n - x_m| \leq (\frac{1}{2})^n + (\frac{1}{2})^m$.

Remark 3.16 If x is a cauchy sequence and q a rational number, the sequence $(x-q)_n = (x_n - q)$ is also Cauchy.

Following [BB85], we define inequality relations between Cauchy sequences and rational numbers.

Definition 3.17 For x a Cauchy sequence and q a rational number, we define

- $x \leq q = \Pi_{n:\mathbb{N}} x_n \leq q + (\frac{1}{2})^n$.
- $x \geq q = \Pi_{n:\mathbb{N}} x_n \geq q - (\frac{1}{2})^n$.

Lemma 3.18 For x a Cauchy sequence and q a rational number, we have $x \leq q \vee x \geq q$.

Proof For rational numbers, we have decidable inequalities, therefore $\geq 0 \vee q \leq 0$. It follows that $\forall (n : \mathbb{N}) \forall (m : \mathbb{N}) q \geq -(\frac{1}{2})^n \vee q \leq (\frac{1}{2})^m$. Now by **??**, we may conclude $(\forall (n : \mathbb{N}) q \geq -(\frac{1}{2})^n) \vee (\forall (m : \mathbb{N}) q \leq (\frac{1}{2})^m)$ as required. □

Definition 3.19 Given two Cauchy sequences $p = (p_n)_{n \in \mathbb{N}}, q = (q_n)_{n \in \mathbb{N}}$, we define the proposition $p \sim_C q$ as

$$p \sim_C q := \forall (n, m : \mathbb{N})((|p_n - q_m| \leq (\frac{1}{2})^n + (\frac{1}{2})^m)) \tag{9}$$

Definition 3.20 The type of **Cauchy reals** is given by the type of Cauchy sequences modulo $\sim_C$.

We claim that the inequality in **??** extends to a well-defined inequality between Cauchy reals and rational numbers.

Furthermore, we claim that $\Pi_{x:\mathbb{R}} \Pi_{q:\mathbb{Q}} x \leq q \vee x \geq q$.

Definition 3.21 A Cauchy sequence in the interval is a Cauchy sequence x such that for any $n : \mathbb{N}$, we have $0 \leq x_n \leq 1$. The interval of Cauchy reals is given by the type of Cauchy sequences in the interval modulo $\sim_C$. We denote it by $[0,1]$.

We want to show that the interval of Cauchy reals is Compact Hausdorff. Informally, to any binary sequence $\alpha : \mathbb{N} \to 2$, we can associate a Cauchy sequence

$$n \mapsto \sum_{i=0}^{n} \frac{\alpha(i)}{2^{i+1}} \tag{10}$$

and we are going to give a closed relation on Cantor space such that two binary sequences are equivalent iff they correspond to the same Cauchy reals. First, we'll need some notation.

Definition 3.22 Given a binary sequence $\alpha : \mathbb{N} \to 2$ and a natural number $n : \mathbb{N}$ we denote $\alpha|_n : \mathbb{N}_{\leq n} \to 2$ for the restriction of α to a finite sequence of length n. We denote $\overline{0}, \overline{1}$ for the binary sequences which are constantly 0 and 1 respectively. We denote $0, 1$ for the sequences of length 1 hitting $0, 1$ respectively. If x is a finite sequence and y is any sequence, denote $x \cdot y$ for their concatenation.

Now we'll give a definition for when two finite binary sequences of length n correspond to real numbers whose distance is $\leq (\frac{1}{2})^n$. Basically, we want for every finite sequence z that $(z \cdot 0 \cdot \overline{1})$ and $(z \cdot 1 \cdot \overline{0})$ are equivalent.

Definition 3.23 Now let $n : \mathbb{N}$ and $x, y : \mathbb{N}_{\leq n} \to 2$ be two sequences of length n. We say x, y are near if we have an $m : \mathbb{N}$ with $m \leq n$ and some $a : \mathbb{N}_{\leq m} \to 2$, such that one of $(a \cdot 0 \cdot \overline{1})|_n, (a \cdot 1 \cdot \overline{0})|_n$ is equal to x and the other is equal to y. We denote $\mathrm{near}_n(x, y)$ if x, y are near. To be precise, we define

$$\mathrm{near}_n(x,y) = \Sigma(m : \mathbb{N}) m \leq n \wedge \Sigma(a : Fin_m \to 2)\Big(\big((x,y) = ((a{\cdot}0{\cdot}\overline{1})|_n, (a{\cdot}1{\cdot}\overline{0})|_n)\big) \bigvee \big((y,x) = ((a{\cdot}0{\cdot}\overline{1})|_n, (a{\cdot}1{\cdot}\overline{0})|_n)\big)\Big) \tag{11}$$

Remark 3.24 Remark that when x, y are near, m and a as above are unique. Thus $\mathrm{near}_n(x, y)$ is a proposotion. Furthermore, to check whether x, y are near, we need only make n comparisons, thus $\mathrm{near}_n(x, y)$ is decidable. Note that in the above definition, we allow $m = n$ and therefore x is near to itself for any finite sequence x. Furthermore, we have defined nearness to be symmetric. However, it is not a transtive relation. After all, the sequence 010 and 011 are near and the sequence 011 and 100 are near, but 010 is not near to 100. This corresponds to the fact that $\frac{1}{4}$ and $\frac{3}{8}$ are distance $\leq (\frac{1}{2})^3$ apart, and so are $\frac{3}{8}$ and $\frac{1}{2}$, but $\frac{1}{4}$ and $\frac{1}{2}$ are not.

Definition 3.25 We define the following relation on Cantor space for $\alpha, \beta : 2^{\mathbb{N}}$.

$$\alpha \sim_t \beta = \forall (n : \mathbb{N}) \mathrm{near}_n(\alpha|_n, \beta|_n) \tag{12}$$

Lemma 3.26 $\sim_t$ is a closed equivalence relation.

Proof Let $\alpha, \beta, \gamma : 2^{\mathbb{N}}$. As the dependent product of propositions is a proposition, $\alpha \sim_t \beta$ is a proposition. Furthermore, the closedness follows from decidability of $\mathrm{near}_n(\alpha|_n, \beta|_n)$. One could define $\gamma(n) = 1$ iff $\mathrm{near}_n(\alpha|_n, \beta|_n)$

As nearness is reflexive and symmetric, so is $\sim_t$.

Now suppose $\alpha \sim_t \beta$ and $\beta \sim_t \gamma$. We claim that $\alpha \sim_t \gamma$.

Let $n : \mathbb{N}$, we need to show that $\mathrm{near}_n(\alpha|_n, \gamma|_n)$. Let (a, m) witness that $\mathrm{near}_n(\alpha|_n, \beta|_n)$. and let (b, k) witness that $\mathrm{near}_n(\beta|_n, \gamma|_n)$ We will make a case distinction on whether one of m, k is equal to n, or both are strictly smaller than n.

- If $m = n$, we have that $\alpha|_n = \beta|_n$, and therefore

  $$\mathrm{near}_n(\beta|_n, \gamma|_n) \leftrightarrow \mathrm{near}_n(\alpha|_n, \gamma|_n) \tag{13}$$

  The above also holds if $k = n$.

- If $m < n$, we have that $\alpha(m + 1) \neq \beta(m + 1)$, thus $\alpha|_l \neq \beta|_l$ for all $l > m$, but we still have $\mathrm{near}_l(\alpha|_l, \beta|_l)$ for these l. Therefore (α, β) or (β, α) must be of the form $(a \cdot 0 \cdot \overline{1}, a \cdot 1 \cdot \overline{0})$. WLOG, we assume $\alpha = a \cdot 0 \cdot \overline{1}$, and thus $\beta = a \cdot 1 \cdot \overline{0}$ (if not, we could do bitflips).

  As $k < n$ also, by the same argument there is some b such that one of $(\beta, \gamma), (\gamma, \beta)$ is equal to $(b \cdot 0 \cdot \overline{1}, b \cdot 1 \cdot \overline{0})$. However, β is also of the form $a \cdot 1 \cdot \overline{0}$, and thus cannot also be of the form $b \cdot 0 \cdot \overline{1}$. Therefore we must have $\beta = b \cdot 1 \cdot \overline{0}$ and $\gamma = b \cdot 0 \cdot \overline{1}$.

  But now $b \cdot 1 \cdot \overline{0} = a \cdot 1 \cdot \overline{0}$, The lengths of a, b cannot be unequal, and by decidablity of natural numbers, a, b have the same length and it follows that $a = b$. Therefore $\alpha = \gamma$, so $\alpha \sim_t \gamma$.

We conclude that $\sim_t$ is a closed equivalence relation. □

Lemma 3.27 b sends $\sim_n$ equivalent binary sequences to $\sim_C$ equivalent Cauchy sequences.

Proof Let α, β be binary sequences. We claim that $|b(\alpha)_n - b(\beta)_n| \leq (\frac{1}{2})^{n+1}$ whenever $\mathrm{near}_n(\alpha, \beta)$. It will follow that if $\alpha \sim_n \beta$, then $b(\alpha) \sim_C b(\beta)$.

Let $n : \mathbb{N}$ and assume $m : \mathbb{N}$ with $m \leq n$ and let z be a sequence of length m such that $\alpha|_n = z \cdot 1 \cdot \overline{0}|_n$ and $\beta|_n = z \cdot 0 \cdot \overline{q}|_n$. then $b(\alpha)_n = \sum_{i \leq m} \frac{z(i)}{2^{i+1}} + (\frac{1}{2})^{m+2}$ and $b(\beta)_n = \sum_{i \leq m} \frac{z(i)}{2^{i+1}} + \sum\limits_{m+2 \leq i \leq n} (\frac{1}{2})^{i+1}$. Thus $b(\alpha)_n - b(\beta)_n = (\frac{1}{2})^{m+2} - \sum\limits_{m+2 \leq i \leq n} (\frac{1}{2})^{i+1} = (\frac{1}{2})^{n+1}$, which is smaller than required. □

Lemma 3.28 Whenever $b(\alpha) \sim_C b(\beta)$, we have $\alpha \sim_n \beta$.

Proof Assume $b(\alpha) \sim_C b(\beta)$. Let $n : \mathbb{N}$. We shall show that $\mathrm{near}_n(\alpha, \beta)$.

As we're only checking finitely many entries, we either have $\alpha|_n = \beta|_n$, or there exists a smallest $m \leq n$ with $\alpha(m) \neq \beta(m)$.

If $\alpha|_n = \beta|_n$, we have $\mathrm{near}_n(\alpha, \beta)$ and are done. WLOG assume $\alpha(m) = 1, \beta(m) = 0$ for m minimal.

Now note that

$$b(\alpha)_{k+1} - b(\beta)_{k+1} = b(\alpha)_k - b(\beta)_k + \frac{\alpha(k+1) - \beta(k+1)}{2^{k+2}}. \tag{14}$$

For $k > m$, we have that

$$|b(\alpha)_k - b(\beta)_k| = |(\frac{1}{2})^{m+1} + \sum_{i=m+1}^{k} \frac{\alpha(i) - \beta(i)}{2^{i+1}}|. \tag{15}$$

Note that the right summand is always $\leq (\frac{1}{2})^{m+1}$. Therefore, we can leave out the absolute value function.

We claim that for every $k \geq m+1$, we have $\alpha(k) = 0, \beta(k) = 1$. We will use induction. Suppose that for every $m < i < j$, we have $\alpha(i) = 0$, and $\beta(i) = 1$. Then

$$b(\alpha)_{j-1} - b(\beta)_{j-1} = (\frac{1}{2})^{m+1} + \sum_{i=m+1}^{j-1} \frac{-1}{2^{i+1}} = (\frac{1}{2})^j \tag{16}$$

- we claim that $\alpha(j) = 0$ Suppose $\alpha(j) = 1$. Then $\alpha(j) - \beta(j) \geq 0$. And for $j+2$, we have that

$$\begin{aligned}
&b(\alpha)_{j+2} - b(\beta)_{j+2} && (17)\\
=&(b(\alpha)_{j-1} - b(\beta)_{j-1}) + \frac{\alpha(j) - \beta(j)}{2^{j+1}} + \frac{\alpha(j+1) - \beta_(j+1)}{2^{j+2}} + \frac{\alpha(j+2) - \beta_(j+2)}{2^{j+3}} && (18)\\
\geq&(\frac{1}{2})^j + 0 + \frac{-1}{2^{j+2}} + \frac{-1}{2^{j+3}} && (19)\\
>&(\frac{1}{2})^{j+1} && (20)
\end{aligned}$$

  which contradicts $b(\alpha) \sim_C b(\beta)$, which would require that $|b(\alpha)_{j+2} - b(\beta_{j+2}| \leq (\frac{12}{\text{)}}^{j+2} + (\frac{1}{2})^{j+2} = (\frac{1}{2})^{j+1}$. Therefore $\alpha(j) \neq 1$, and thus $\alpha(j) = 0$.
- We also claim that $\beta(i) = 1$. If $\beta(i) = 0$, we also have $\alpha(j) - \beta(j) \geq 0$, and the rest of the proof is similar as above. □

Lemma 3.29 The map $b : 2^{\mathbb{N}} \to [0, 1]$ is surjective.

Proof First, suppose we have a function $d : \Pi_{x:\mathbb{R}} \Pi_{q:\mathbb{Q}} (x \leq q + x \geq q)$ Then we could recursively define

$$\alpha(n) = \begin{cases} 0 \text{ if } d(x - \sum_{i<n} \frac{\alpha(i)}{2^{i+1}}, \frac{1}{2^{n+1}}) = inl(\cdot) \\ 1 \text{ otherwise} \end{cases}$$

Note that

$$\alpha(n) = \begin{cases} 0 \text{ if } d(x - b(\alpha)_{n-1}, \frac{1}{2^{n+1}}) = inl(\cdot) \\ 1 \text{ otherwise} \end{cases}$$

We'll show by induction that $b(\alpha)_n \leq x$ for every $n : \mathbb{N}$. First $b(\alpha)_0 = 0 \leq x$. Assuming, $b(\alpha)_k \leq x$, for $b(\alpha)_{k+1}$, there are two cases:

- if $d(x - b(\alpha)_k, \frac{1}{2^{n+1}}) = inl(\cdot)$, then $b(\alpha)_{k+1} = b(\alpha)_k$, which is $\leq x$ by induction hypothesis.
- Otherwise, $x - b(\alpha)_k \geq (\frac{1}{2})^{k+1}$ So $x - b(\alpha)_k - (\frac{1}{2})^{k+1} \geq 0$, and $b(\alpha)_{k+1} = b(\alpha)_k + (\frac{1}{2})^{k+1}$. So $x - b(\alpha)_{k+1} \geq 0$, and $b(\alpha)_{k+1} \leq x$ as required.

So by induction $b(\alpha)_n \leq x$ for every $n : \mathbb{N}$. Therefore, $|x - b(\alpha)_n| = x - b(\alpha)_n$.

We shall also show by induction that $x - b(\alpha)_n \leq (\frac{1}{2})^{n+1}$ for every natural number $n : \mathbb{N}$. For $n = 0$, this follows from the assumption that $x \leq 1$. Suppose that $x - b(\alpha)_k \leq (\frac{1}{2})^{k+1}$. We make a case distinction on the form of $d(x - b(\alpha)_k, (\frac{1}{2})^{k+2})$.

- If $d(x - b(\alpha)_k, (\frac{1}{2})^{k+2}) = inl(\cdot)$, then $x - b(\alpha)_k \leq (\frac{1}{2})^{k+2}$, and $b(\alpha)_{k+1} = b(\alpha)_k$, and $x - b(\alpha)_{k+1} \leq (\frac{1}{2})^{k+2}$ as well, as required.
- Otherwise, we must have $x - b(\alpha)_k \geq (\frac{1}{2})^{k+2}$, and $b(\alpha)_{k+1} = b(\alpha)_k + (\frac{1}{2})^{k+1}$. By induction hypothesis, we have $x - b(\alpha)_k \leq (\frac{1}{2})^{k+1}$. Thus

$$x - b(\alpha)_{k+1} = x - b(\alpha)_k - (\frac{1}{2})^{k+1} \leq (\frac{1}{2})^{k+1} - (\frac{1}{2})^{k+2} = (\frac{1}{2})^{k+2} \tag{21}$$

  as required.

By induction, we conclude that $|b(\alpha)_n - x| \leq (\frac{1}{2})^{n+1}$ for every $n : \mathbb{N}$. Therefore $b(\alpha)$ converges to x.

We may conclude that $\Pi_{x:[0,1]}\Pi_{q:\mathbb{Q}}(x \leq q + x \geq q)$ implies that we can give for each $x : [0,1]$ a binary sequence α with $b(\alpha) = x$. As we have the propositional trunctation of the premise by Lemma 3.18, we may conclude that for each $x : [0,1]$ there merely exists α with $b(\alpha) = x$. Therefore b is surjective. $\square$

Theorem 3.30
The interval of Cauchy reals is isomorphic to $2^{\mathbb{N}}/\sim_t$.

Proof This follows from the fact that $b : 2^{\mathbb{N}}$ is such that $\alpha \sim_n \beta$ iff $b(\alpha) \sim_t b(\beta)$. and for every Cauchy real, there is a binary sequence being sent to it, so the composition of b and the quotient from Caucy sequences to Cauchy real is a surjection. $\square$

Corollary 3.31 The interval is compact Hausdorff.

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