A Foundation for Synthetic Stone Duality

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January 9, 2025

Abstract

The language of homotopy type theory has proved to be appropriate as an internal language for various higher toposes, for example with Synthetic Algebraic Geometry for the Zariski topos. In this paper we apply such techniques to the higher topos corresponding to the light condensed sets of Dustin Clausen and Peter Scholze. This seems to be an appropriate setting to develop synthetic topology, similar to the work of Martín Escardó. To reason internally about light condensed sets, we use homotopy type theory extended with 4 axioms. Our axioms are strong enough to prove Markov's principle, LLPO and the negation of WLPO. We also define a type of open propositions, inducing a topology on any type. This leads to a synthetic topological study of (second countable) Stone and compact Hausdorff spaces. Indeed all functions are continuous in the sense that they respect this induced topology, and this topology is as expected for these classes of types. For example, any map from the unit interval to itself is continuous in the usual epsilon-delta sense. We also use the synthetic homotopy theory given by the higher types of homotopy type theory to define and work with cohomology. As an application, we prove Brouwer's fixed-point theorem internally.

Acknowledgements

The idea to use the topological characterization of stone spaces as totally disconnected, compact Hausdorff spaces to prove Theorem 4.3.7 was explained to us by Martín Escardó. We profited a lot from a discussion with Reid Barton and Johann Commelin. David Wärn noticed that Markov's principle (Corollary 1.4.3) holds. At TYPES 2024, we had an interesting discussion with Bas Spitters on the topic of the article. Work on this article was supported by the ForCUTT project, ERC advanced grant 101053291.

Introduction

The language of homotopy type theory is a dependent type theory enriched with the univalence axiom and higher inductive types. It has proven exceptionnally well-suited to develop homotopy theory in a synthetic way [Pro13]. It also provides the precision needed to analyze categorical models of type theory [Wei24]. Moreover, the arguments in this language can be rather directly represented in proof assistants. We use homotopy type theory to give a synthetic development of topology, which is analogous to the work on synthetic algebraic geometry [CCH23].

We introduce four axioms which seem sufficient for expressing and proving basic notions of topology, based on the light condensed sets approach, introduced in [CS24]. Interestingly, this development establishes strong connections with constructive mathematics [BB85], particularly constructive reverse mathematics [Ish06; Die18]. Several of Brouwer's principles, such that any real function on the unit interval is continuous, or the celebrated fan theorem, are consequences of this system of axioms. Furthermore, we can also prove principles that are not intuitionistically valid, such as Markov's Principle, or even the so-called Lesser Limited Principle of Omniscience, a principle well studied in constructive reverse mathematics, which is *not* valid effectively.

This development also aligns closely with the program of Synthetic Topology [Esc04; Leš21]: there is a dominance of open propositions, providing any type with an intrinsic topology, and we capture in this way synthetically the notion of (second-countable) compact Hausdorff spaces. While working on this axiom system, we learnt about the related work [BC], which provides a different axiomatisation at the set level. We show that some of their axioms are consequences of our axiom system. In particular, we can introduce in our setting a notion of "Overtly Discrete" spaces, dual in some way to the notion of compact Hausforff spaces, like in Synthetic Topology[1].

A central theme of homotopy type theory is that the notion of *type* is more general than the notion of *set*. We illustrate this theme here as well: we can form in our setting the types of Stone spaces and of compact Hausdorff spaces (types which don't form a set but a groupoid), and show these types are closed under sigma types. It would be impossible to formulate such properties in the setting of simple type or set theory. Additionally, leveraging the elegant definition of cohomology groups in homotopy type theory [Pro13], which relies on higher types that are not sets, we prove, in a purely axiomatic way, a special case of a theorem of Dyckhoff [Dyc76], describing the cohomology of compact Hausdorff spaces. This characterisation also supports a type-theoretic proof of Brouwer's fixed point theorem, similar to the proof in [Shu18]. In our setting the theorem can be formulated in the usual way, and not in an approximated form.

It is important to stress that what we capture in this axiomatic way are the properties of light condensed sets that are *internally* valid. David Wärn [Wär24] has proved that an important property of abelian groups in the setting of light condensed sets, is *not* valid internally and thus cannot be proved in this axiomatic context. We believe however that our axiom system can be convenient for proving the results that are internally valid, as we hope is illustrated by the present paper. We also conjecture that the present axiom system is actually *complete* for the properties that are internally valid. Finally, we think that this system can be justified in a constructive metatheory using the work [CRS21].

1 Stone duality

1.1 Preliminaries

Remark 1.1.1 For X any type, a subtype U is a family of propositions over X. We write $U \subseteq X$. If X is a set, we call U a subset. Given $x : X$ we sometimes write $x \in U$ instead of $U(x)$. For subtypes $A, B \subseteq X$, we write $A \subseteq B$ for pointwise implication. We will freely switch between subtypes $U \subseteq X$ and the corresponding embeddings $\sum_{x:X} U(x) \hookrightarrow X$. In particular, if we write $x : U$ we mean $x : X$ such that $U(x)$.

Definition 1.1.2 A type is countable if and only if it is merely equal to some decidable subset of $\mathbb{N}$.

Definition 1.1.3 For I a set we write $2[I]$ for the free Boolean algebra on I. A Boolean algebra B is countably presented, if there exist countable sets I, J, generators $g_i : C,\ i \in I$ and relations $f_j : 2[I],\ j \in J$ such that g induces an equivalence between B and $2[I]/(f_j)_{j:J}$.

Remark 1.1.4 Any countably presented algebra is merely of the form $2[\mathbb{N}]/(r_n)_{n:\mathbb{N}}$.

Remark 1.1.5 We denote the type of countably presented Boolean algebras by Boole. This type does not depend on a choice of universe. Moreover Boole has a natural category structure.

Example 1.1.6 If both the set of generators and relations are empty, we have the Boolean algebra 2. Its underlying set is $\{0, 1\}$ and $0 \neq_2 1$. 2 is initial in Boole.

Definition 1.1.7 For B a countably presented Boolean algebra, we define the spectrum $Sp(B)$ as the set of Boolean morphisms from B to 2. Any type which is merely equivalent to a type of the form $Sp(B)$ is called a Stone space.

Example 1.1.8

(i) There is only one Boolean morphism from 2 to 2, thus $Sp(2)$ is the singleton type $\top$.

(ii) The tivial Boolean algebra is given by $2/(1)$. We have $0 = 1$ in the trivial Boolean algebra, so there cannot be a map from it into 2 preserving both 0 and 1. So the corresponding Stone space is the empty type $\bot$.

(iii) The type $Sp(2[\mathbb{N}])$ is called the Cantor space. It is equivalent to the set of binary sequences $2^{\mathbb{N}}$. If $\alpha : Sp(2[\mathbb{N}])$ and $n : \mathbb{N}$ we write α_n for $\alpha(g_n)$.

[1] We actually have a derivation of their "directed univalence", but this will be presented in a following paper.

(iv) We denote by B_∞ the Boolean algebra generated by $(g_n)_{n:\mathbb{N}}$ quotiented by the relations $g_m \wedge g_n = 0$ for $n \neq m$. A morphism $B_\infty \to 2$ corresponds to a function $\mathbb{N} \to 2$ that hits 1 at most once. We denote $Sp(B_\infty)$ by $\mathbb{N}_\infty$. For $\alpha : \mathbb{N}_\infty$ and $n : \mathbb{N}$ we write α_n for $\alpha(g_n)$. By conjunctive normal form, any element of B_∞ can be written uniquely as $\bigvee_{i:I} g_n$ or as $\bigwedge_{i:I} \neg g_n$ for some finite $I \subseteq \mathbb{N}$.

Lemma 1.1.9 For $\alpha : 2^{\mathbb{N}}$, we have an equivalence of propositions:

$$(\forall_{n:\mathbb{N}} \alpha_n = 0) \leftrightarrow Sp(2/(\alpha_n)_{n:\mathbb{N}}).$$

Proof There is only one boolean morphism $x : 2 \to 2$, and it satisfies $x(\alpha_n) = 0$ for all $n : \mathbb{N}$ if and only if $\alpha_n = 0$ for all $n : \mathbb{N}$. $\square$

1.2 Axioms

Axiom 1 (Stone duality)
For any B : Boole, the evaluation map $B \to 2^{Sp(B)}$ is an isomorphism.

Axiom 2 (Surjections are formal surjections)
For $g : B \to C$ a map in Boole, g is injective if and only if $(-) \circ g : Sp(C) \to Sp(B)$ is surjective.

Axiom 3 (Local choice)
Whenever we have B : Boole, and some type family P over $Sp(B)$ with $\Pi_{s:Sp(B)} \|P(s)\|$, then there merely exists some C : Boole and surjection $q : Sp(C) \to Sp(B)$ with $\Pi_{t:Sp(C)} P(q(t))$.

Axiom 4 (Dependent choice)
Given types $(E_n)_{n:\mathbb{N}}$ with for all $n : \mathbb{N}$ a surjection $E_{n+1} \twoheadrightarrow E_n$, the projection from the sequential limit $\lim_k E_k$ to E_0 is surjective.

1.3 Anti-equivalence of Boole and Stone

By Axiom 1, Sp is an embedding of Boole into any universe of types. We denote its image by Stone.

Remark 1.3.1 Stone spaces will take over the role of affine scheme from [CCH23], and we repeat some results here. Analogously to Lemma 3.1.2 of [CCH23], for X Stone, Stone duality tells us that $X = Sp(2^X)$. Proposition 2.2.1 of [CCH23] now says that Sp gives a natural equivalence

$$Hom_{\mathsf{Boole}}(A, B) = (Sp(B) \to Sp(A))$$

Stone also has a natural category structure. By the above and Lemma 9.4.5 of [Pro13], the map Sp defines a dual equivalence of categories between Boole and Stone. In particular the spectrum of any colimit in Boole is the limit of the spectrum of the opposite diagram.

Remark 1.3.2 Axiom 3 can also be formulated as follows: whenever we have S : Stone, E, F arbitrary types, a map $f : S \to F$ and a surjection $e : E \twoheadrightarrow F$, there exists a Stone space T, a surjective map $T \twoheadrightarrow S$ and an arrow $T \to E$ making the following diagram commute:

$$\begin{tikzcd} T \arrow[r, dashed] \arrow[d, dashed, two heads] & E \arrow[d, two heads, "e"] \\ S \arrow[r, "f"'] & F \end{tikzcd}$$

Lemma 1.3.3 For B : Boole, we have $0 =_B 1$ if and only if $\neg Sp(B)$.

Proof If $0 =_B 1$, there is no map $B \to 2$ preserving both 0 and 1, thus $\neg Sp(B)$. Conversely, if $\neg Sp(B)$, then $Sp(B)$ equals $\perp$, the spectrum of the trivial Boolean algebra. As Sp is an embedding, B is equivalent to the trivial Boolean algebra, hence $0 =_B 1$. $\square$

Corollary 1.3.4 For S : Stone, we have that $\neg\neg S \to \|S\|$

Proof Let B : Boole and suppose $\neg\neg Sp(B)$. By Lemma 1.3.3 we have that $0 \neq_B 1$, therefore the morphism $2 \to B$ is injective. By Axiom 2 the map $Sp(B) \to Sp(2)$ is surjective, thus $Sp(B)$ is merely inhabited. $\square$

1.4 Principles of omniscience

In constructive mathematics, we do not assume the law of excluded middle (LEM). There are some principles called principles of omniscience that are weaker than LEM, which can be used to describe how close a logical system is to satisfying LEM. References on these principles include [Die18; Ish06]. In this section, we will show that two of them (MP and LLPO) hold, and one (WLPO) fails in our system.

Theorem 1.4.1 (The negation of the weak lesser principle of omniscience ($\neg$WLPO))

$$\neg\forall_{\alpha:2^{\mathbb{N}}}((\forall_{n:\mathbb{N}}\alpha_n = 0) \vee \neg(\forall_{n:\mathbb{N}}\alpha_n = 0))$$

Proof Assume $f : 2^{\mathbb{N}} \to 2$ such that $f(\alpha) = 0$ if and only if $\forall_{n:\mathbb{N}}\alpha_n = 0$. By Axiom 1, there is some $c : 2[\mathbb{N}]$ with $f(\alpha) = 0 \leftrightarrow \alpha(c) = 0$. There exists $k : \mathbb{N}$ such that c is expressed the generators $(g_n)_{n\leq k}$. Now consider $\beta, \gamma : 2^{\mathbb{N}}$ given by $\beta(g_n) = 0$ for all $n : \mathbb{N}$ and $\gamma(g_n) = 0$ if and only if $n \leq k$. As β, γ are equal on $(g_n)_{n\leq k}$, we have $\beta(c) = \gamma(c)$. However, $f(\beta) = 0$ and $f(\gamma) = 1$, giving a contradiction. $\square$

Theorem 1.4.2
For $\alpha : \mathbb{N}_\infty$, we have that

$$(\neg(\forall_{n:\mathbb{N}}\alpha_n = 0)) \to \Sigma_{n:\mathbb{N}}\alpha_n = 1$$

Proof By Lemma 1.1.9, we have that $\neg(\forall_{n:\mathbb{N}}\alpha_n = 0)$ implies that $Sp(2/(\alpha_n)_{n:\mathbb{N}}$ is empty. Hence $2/(\alpha_n)_{n:\mathbb{N}}$ is trivial by Lemma 1.3.3. Then there exists $k : \mathbb{N}$ such that $\bigvee_{i\leq k} \alpha_i = 1$. As $\alpha_i = 1$ for at most one $i : \mathbb{N}$, there exists an unique $n : \mathbb{N}$ with $\alpha_n = 1$. $\square$

Corollary 1.4.3 (Markov's principle (MP)) For $\alpha : 2^{\mathbb{N}}$, we have that

$$(\neg(\forall_{n:\mathbb{N}}\alpha_n = 0)) \to \Sigma_{n:\mathbb{N}}\alpha_n = 1$$

Proof Given $\alpha : 2^{\mathbb{N}}$, consider the sequence $\alpha' : \mathbb{N}_\infty$ satisfying $\alpha'_n = 1$ if and only if n is minimal with $\alpha_n = 1$. Then apply the above theorem. $\square$

Theorem 1.4.4 (The lesser limited principle of omniscience (LLPO))
For $\alpha : \mathbb{N}_\infty$, we have:

$$\forall_{k:\mathbb{N}}\alpha_{2k} = 0 \vee \forall_{k:\mathbb{N}}\alpha_{2k+1} = 0$$

Proof Define $f : B_\infty \to B_\infty \times B_\infty$ on generators as follows:

$$f(g_n) = \begin{cases} (g_k, 0) \text{ if } n = 2k \\ (0, g_k) \text{ if } n = 2k + 1 \end{cases}$$

Note that f is well-defined as map in Boole as $f(g_n) \wedge f(g_m) = 0$ whenever $m \neq n$. We claim that f is injective. If $I \subseteq \mathbb{N}$, write $I_0 = \{k \mid 2k \in I\}, I_1 = \{k \mid 2k + 1 \in I\}$. Recall that any $x : B_\infty$ is of the form $\bigvee_{i\in I} g_i$ or $\bigwedge_{i\in I} \neg g_i$ for some finite set I.

- If $x = \bigvee_{i\in I} g_i$, then $f(x) = (\bigvee_{i\in I_0} g_i, \bigvee_{i\in I_1} g_i)$. So if $f(x) = 0$, then $I_0 = I_1 = I = \emptyset$ and $x = 0$.
- Suppose $x = \bigwedge_{i\in I} \neg g_i$. Then $f(x) = (\bigwedge_{i\in I_0} \neg g_i, \bigwedge_{i\in I_1} \neg g_i)$, so $f(x) \neq 0$.

By Axiom 2, f corresponds to a surjection $s : \mathbb{N}_\infty + \mathbb{N}_\infty \to \mathbb{N}_\infty$. Thus for $\alpha : \mathbb{N}_\infty$, there exists some $x : \mathbb{N}_\infty + \mathbb{N}_\infty$ such that $s(x) = \alpha$. If $x = inl(\beta)$, for any $k : \mathbb{N}$, we have that

$$\alpha_{2k+1} = s(x)_{2k+1} = x(f(g_{2k+1})) = inl(\beta)(0, g_k) = \beta(0) = 0.$$

Similarly, if $x = inr(\beta)$, we have $\alpha_{2k} = 0$ for all $k : \mathbb{N}$. $\square$

The surjection $s : \mathbb{N}_\infty + \mathbb{N}_\infty \to \mathbb{N}_\infty$ as above does not have a section as the following shows:

Lemma 1.4.5 The function f defined above does not have a retraction.

Proof Suppose $r : B_\infty \times B_\infty \to B_\infty$ is a retraction of f. Then $r(0, g_k) = g_{2k+1}$ and $r(g_k, 0) = g_{2k}$. Note that $r(0, 1) \geq r(0, g_k) = g_{2k+1}$ for all $k : \mathbb{N}$. As a consequence, $r(0, 1)$ is of the form $\bigwedge_{i\in I} \neg g_i$ for some finite set I. By similar reasoning so is $r(1, 0)$. But this contradicts:

$$r(0, 1) \wedge r(1, 0) = r((1, 0) \wedge (0, 1)) = r(0, 0) = 0.$$

Thus no retraction exists. $\square$

1.5 Open and closed propositions

In this section we will introduce a topology on the type of propositions, and study their logical properties. We think of open and closed propositions respectively as countable disjunctions and conjunctions of decidable propositions. Such a definition is universe-independent, and can be made internally.

Definition 1.5.1 A proposition P is open (resp. closed) if there exists some $\alpha : 2^{\mathbb{N}}$ such that $P \leftrightarrow \exists_{n:\mathbb{N}}\alpha_n = 0$ (resp. $P \leftrightarrow \forall_{n:\mathbb{N}}\alpha_n = 0$). We denote by Open and Closed the types of open and closed propositions.

Remark 1.5.2 The negation of an open proposition is closed, and by MP (Corollary 1.4.3), the negation of a closed proposition is open and both open, closed propositions are $\neg\neg$-stable. By $\neg$WLPO (Theorem 1.4.1), not every closed proposition is decidable. Therefore, not every open proposition is decidable. Every decidable proposition is both open and closed.

Lemma 1.5.3 We have the following results on open and closed propositions:

- Closed propositions are closed under finite disjunctions.
- Closed propositions are closed under countable conjunctions.
- Open propositions are closed under finite conjunctions.
- Open propositions are closed under countable disjunctions.

Proof By Proposition 1.4.1 of [Die18], LLPO(Theorem 1.4.4) is equivalent to the statement that the disjunction of two closed propositions are closed. The other statements have similar proofs, and we only present the proof that closed propositions are closed under countable conjunctions. Let $(P_n)_{n:\mathbb{N}}$ be a countable family of closed propositions. By countable choice, for each $n : \mathbb{N}$ we have an $\alpha_n : 2^{\mathbb{N}}$ such that $P_n \leftrightarrow \forall_{m:\mathbb{N}}\alpha_{n,m} = 0$. Consider a surjection $s : \mathbb{N} \twoheadrightarrow \mathbb{N} \times \mathbb{N}$, and let $\beta_k = \alpha_{s(k)}$. Note that $\forall_{k:\mathbb{N}}\beta_k = 0$ if and only if $\forall_{n:\mathbb{N}}P_n$. $\square$

We will use the above properties silently from now on.

Corollary 1.5.4 If a proposition is both open and closed, it is decidable.

Proof If P is open and closed, $P \vee \neg P$ is open, hence $\neg\neg$-stable and provable. $\square$

Lemma 1.5.5 For $(P_n)_{n:\mathbb{N}}$ a sequence of closed propositions, we have $\neg\forall_{n:\mathbb{N}}P_n \leftrightarrow \exists_{n:\mathbb{N}}\neg P_n$.

Proof Both $\neg\forall_{n:\mathbb{N}}P_n$ and $\exists_{n:\mathbb{N}}\neg P_n$ are open, hence $\neg\neg$-stable. The equivalence follows. $\square$

Lemma 1.5.6 If P is open and Q is closed then $P \to Q$ is closed. If P is closed and Q open, then $P \to Q$ is open.

Proof Note that $\neg P \vee Q$ is closed. Using $\neg\neg$-stability we can show $(P \to Q) \leftrightarrow (\neg P \vee Q)$. The other proof is similar. $\square$

1.6 Types as spaces

The subobject Open of the type of propositions induces a topology on every type. This is the viewpoint taken in synthetic topology. We will follow the terminology of [Esc04; Leš21].

Definition 1.6.1 Let T be a type, and let $A \subseteq T$ be a subtype. We call $A \subseteq T$ open (resp. closed) if $A(t)$ is open (resp. closed) for all $t : T$.

Remark 1.6.2 It follows immediately that the pre-image of an open by any map of types is open, so that any map is continuous. In Theorem 3.3.1, we shall see that the resulting topology is as expected for second countable Stone spaces. In Lemma 5.0.8, we shall see that the same holds for the unit interval.

2 Overtly discrete spaces

Definition 2.0.1 We call a type overtly discrete if it is a sequential colimit of finite sets.

Remark 2.0.2 It follows from Corollary 7.7 of [SDR20] that overtly discrete types are sets, and that the colimit can be defined as in set theory. The type of overtly discrete types is independent on a choice of universe, so we can write ODisc for this type.

Using dependent choice, we have the following results:

Lemma 2.0.3 A map between overtly discrete sets is a sequential colimit of maps between finite sets.

Lemma 2.0.4 For $f : A \to B$ a sequential colimit of maps of finite sets $f_n : A_n \to B_n$, we have that the factorisation $A \twoheadrightarrow Im(f) \hookrightarrow B$ is the sequential colimit of the factorisations $A_n \twoheadrightarrow Im(f_n) \hookrightarrow B_n$.

Corollary 2.0.5 An injective (resp. surjective) map between overtly discrete types is a sequential colimit of injective (resp. surjective) maps between finite sets.

2.1 Closure properties of ODisc

We can get the following result using Lemma 2.0.3 and dependent choice.

Lemma 2.1.1 Overtly discrete types are closed under sequential colimits.

We have that Σ-types, identity types and propositional truncation commutes with sequential colimits (Theorem 5.1, Theorem 7.4 and Corollary 7.7 in [SDR20]). Then by closure of finite sets under these constructors, we can get the following:

Lemma 2.1.2 Overtly discrete types are closed under Σ-type, identity type and propositional truncation.

2.2 Open and ODisc

Lemma 2.2.1 A proposition is open if and only if it is overtly discrete.

Proof If P is overtly discrete, then $P \leftrightarrow \exists_{n:\mathbb{N}} \|F_n\|$ with F_n finite sets. But a finite set being merely inhabited is decidable, hence P is a countable disjunction of decidable propositions, hence open. Suppose $P \leftrightarrow \exists_{n:\mathbb{N}} \alpha_n = 1$. Let $P_n = \exists_{n \leq k}(\alpha_n = 1)$, which is a decidable proposition, hence a finite set. Then the colimit of P_n is P. □

Corollary 2.2.2 Open propositions are closed under sigma types.

Corollary 2.2.3 (transitivity of openness) Let T be a type, let $V \subseteq T$ open and let $W \subseteq V$ open. Then $W \subseteq T$ is open as well.

Remark 2.2.4 It follows from Proposition 2.25 of [Leš21] that Open is a dominance in the setting of synthetic topology.

Lemma 2.2.5 A type B is overtly discrete if and only if it merely is the quotient of a countable set by an open equivalence relation.

Proof If $B : \mathsf{ODisc}$ is the sequential colimit of finite sets B_n, then B is an open quotient of $(\Sigma_{n:\mathbb{N}} B_n)$. Conversely, assume $B = D/R$ with $D \subseteq \mathbb{N}$ decidable and R open. By dependent choice we get $\alpha : D \to D \to 2^{\mathbb{N}}$ such that $R(x, y) \leftrightarrow \exists_{k:\mathbb{N}} \alpha_{x,y}(k) = 1$. Define $D_n = (D \cap \mathbb{N}_{\leq n})$, and $R_n : D_n \to D_n \to 2$ so that $R_n(x, y)$ is the equivalence relation generated by the relation $\exists_{k \leq n} \alpha_{x,y}(k) = 1$. Then the $B_n = D_n/R_n$ are finite sets, and have colimit B. □

2.3 Relating ODisc and Boole

Lemma 2.3.1 Every countably presented Boolean algebra is merely a sequential colimit of finite Boolean algebras.

Proof Consider a countably presented Boolean algebra of the form $B = 2[\mathbb{N}]/(r_n)_{n:\mathbb{N}}$. For each $n : \mathbb{N}$, let G_n be the union of $\{g_i \mid i \leq n\}$ and the finite set of generators occurring in r_i for some $i \leq n$. Denote $B_n = 2[G_n]/(r_i)_{i\leq n}$. Each B_n is a finite Boolean algebra, and there are canonical maps $B_n \to B_{n+1}$. Then B is the colimit of this sequence. □

Corollary 2.3.2 A Boolean algebra B is overtly discrete if and only if it is countably presented.

Proof Assume $B : \mathsf{ODisc}$. By Lemma 2.2.5, we get a surjection $\mathbb{N} \twoheadrightarrow B$ and that B has open equality. Consider the induced surjective morphism $f : 2[\mathbb{N}] \twoheadrightarrow B$. By countable choice, we get for each $b : 2[\mathbb{N}]$ a sequence $\alpha_b : 2^{\mathbb{N}}$ such that $(f(b) = 0) \leftrightarrow \exists_{k:\mathbb{N}}(\alpha_b(k) = 1)$. Consider $r : 2[\mathbb{N}] \to \mathbb{N} \to 2[\mathbb{N}]$ given by

$$r(b,k) = \begin{cases} b & \text{if } \alpha_b(k) = 1 \\ 0 & \text{if } \alpha_b(k) = 0 \end{cases}$$

Then $B = 2[\mathbb{N}]/(r(b,k))_{b:2^{\mathbb{N}},k:\mathbb{N}}$. Lemma 2.3.1 gives the converse. □

Remark 2.3.3 By Lemma 2.1.2 and Corollary 2.3.2, it follows that any $g : B \to C$ in Boole has an overtly discrete kernel. As a consequence, the kernel is enumerable and $B/Ker(g)$ is in Boole. By uniqueness of epi-mono factorizations and Axiom 2, the factorization $B \twoheadrightarrow B/Ker(g) \hookrightarrow C$ corresponds to $Sp(C) \twoheadrightarrow Sp(B/Ker(g)) \hookrightarrow Sp(B)$.

Remark 2.3.4 Similarly to Lemma 2.0.3 and Lemma 2.0.4 a map (resp. surjection, injection) in Boole is a sequential colimit of maps (resp. surjections, injections) between finite Boolean algebras.

3 Stone spaces

3.1 Stone spaces as profinite sets

Here we present Stone spaces as sequential limits of finite sets. This is the perspective taken in Condensed Mathematics [Sch19; Ásg21; CS24]. Some of the results in this section are specific versions of the axioms used in [BC]. A full generalization is part of future work.

Lemma 3.1.1 Any $S : \mathsf{Stone}$ is merely a sequential limit of finite sets.

Proof Assume $B : \mathsf{Boole}$. By Remark 1.3.1 and Lemma 2.3.1, we have that $Sp(B)$ is the sequential limit of the $Sp(B_n)$, which are finite sets. □

Lemma 3.1.2 A sequential limit of finite sets is a Stone space.

Proof By Remark 1.3.1 and Lemma 2.1.1, we have that Stone is closed under sequential limits, and finite sets are Stone. □

Corollary 3.1.3 Stone spaces are stable under finite limits.

Remark 3.1.4 By Remark 2.3.4 and Axiom 2, maps (resp. surjections, injections) of Stone spaces are sequential limits of maps (resp. surjections, injections) of finite sets.

Lemma 3.1.5 For $(S_n)_{n:\mathbb{N}}$ a sequence of finite types with $S = \lim_n S_n$ and $k : \mathbb{N}$, we have that $\mathrm{Fin}(k)^S$ is the sequential colimit of $\mathrm{Fin}(k)^{S_n}$.

Proof By Remark 1.3.1 we have $\mathrm{Fin}(k)^S = \mathrm{Hom}(2^k, 2^S)$. Since 2^k is finite, we have that $\mathrm{Hom}(2^k, \_)$ commutes with sequential colimits, therefore $\mathrm{Hom}(2^k, 2^S)$ is the colimit of $\mathrm{Hom}(2^k, 2^{S_n})$. By applying Remark 1.3.1 again, the latter type is $\mathrm{Fin}(k)^{S_n}$.

Lemma 3.1.6 For $S : \mathsf{Stone}$ and $f : S \to \mathbb{N}$, there exists some $k : \mathbb{N}$ such that f factors through $\mathrm{Fin}(k)$.

Proof For each $n : \mathbb{N}$, the fiber of f over n is a decidable subset $f_n : S \to 2$. We must have that $Sp(2^S/(f_n)_{n:\mathbb{N}}) = \bot$, hence there exists some $k : \mathbb{N}$ with $\bigvee_{n \leq k} f_n =_{2^S} 1$. It follows that $f(s) \leq k$ for all $s : S$ as required. $\square$

Corollary 3.1.7 For $(S_n)_{n:\mathbb{N}}$ a sequence of finite types with $S = \lim_n S_n$, we have that $\mathbb{N}^S$ is the sequential colimit of $\mathbb{N}^{S_n}$.

Proof By Lemma 3.1.6 we have that $\mathbb{N}^S$ is the sequential colimit of $\mathrm{Fin}(k)^S$. By Lemma 3.1.5, $\mathrm{Fin}(k)^S$ is the sequential colimit of the $\mathrm{Fin}(k)^{S_n}$ and we can swap the sequential colimits to conclude. $\square$

3.2 Closed and Stone

Corollary 3.2.1 For all $S : \mathsf{Stone}$, the proposition $\|S\|$ is closed.

Proof By Lemma 1.3.3, $\neg S$ is equivalent to $0 =_{2^S} 1$, which is open by Lemma 2.3.1 and Lemma 2.2.5. Hence $\neg\neg S$ is a closed proposition, and by Corollary 1.3.4, so is $\|S\|$. $\square$

Corollary 3.2.2 A proposition P is closed if and only if it is a Stone space.

Proof By the above, if S is both a Stone space and a proposition, it is closed. By Lemma 1.1.9, any closed proposition is Stone. $\square$

Lemma 3.2.3 For all $S : \mathsf{Stone}$ and $s, t : S$, the proposition $s = t$ is closed.

Proof Suppose $S = Sp(B)$ and let G be a countable set of generators for B. Then $s = t$ if and only if $s(g) = t(g)$ for all $g : G$. So $s = t$ is a countable conjunction of decidable propositions, hence closed. $\square$

3.3 The topology on Stone spaces

Theorem 3.3.1
Let $A \subseteq S$ be a subset of a Stone space. The following are equivalent:

(i) There exists a map $\alpha : S \to 2^{\mathbb{N}}$ such that $A(x) \leftrightarrow \forall_{n:\mathbb{N}} \alpha_{x,n} = 0$ for any $x : S$.

(ii) There exists a family $(D_n)_{n:\mathbb{N}}$ of decidable subsets of S such that $A = \bigcap_{n:\mathbb{N}} D_n$.

(iii) There exists a Stone space T and some embedding $T \to S$ which image is A

(iv) There exists a Stone space T and some map $T \to S$ which image is A.

(v) A is closed.

Proof

- $(i) \leftrightarrow (ii)$. D_n and α can be defined from each other by $D_n(x) \leftrightarrow (\alpha_{x,n} = 0)$. Then observe that

$$x \in \bigcap_{n:\mathbb{N}} D_n \leftrightarrow \forall_{n:\mathbb{N}}(\alpha_{x,n} = 0)$$

- $(ii) \to (iii)$. Let $S = Sp(B)$. By Axiom 1, we have $(d_n)_{n:\mathbb{N}}$ in B such that $D_n = \{x : S \mid x(d_n) = 0\}$. Let $C = B/(d_n)_{n:\mathbb{N}}$. Then $Sp(C) \to S$ is as desired because:

$$Sp(C) = \{x : S \mid \forall_{n:\mathbb{N}} x(d_n) = 0\} = \bigcap_{n:\mathbb{N}} D_n.$$

- $(iii) \to (iv)$. Immediate.
- $(iv) \to (ii)$. Assume $f : T \to S$ corresponds to $g : B \to C$ in Boole. By Remark 2.3.3, $f(T) = Sp(B/Ker(g))$, and there is a surjection $d : \mathbb{N} \to Ker(g)$. Denote by D_n the decidable subset of S corresponding to d_n. Then we have that $Sp(B/Ker(g)) = \bigcap_{n:\mathbb{N}} D_n$.
- $(i) \to (v)$. By definition.

- $(v) \to (iv)$. We have a surjection $2^{\mathbb{N}} \to \mathsf{Closed}$ defined by $\alpha \mapsto \forall_{n:\mathbb{N}}\alpha_n = 0$. Remark 1.3.2 gives us that there merely exists $T, e, \beta.$ as follows:

$$\begin{tikzcd} T \arrow[r, "\beta"] \arrow[d, two heads, "e"'] & 2^{\mathbb{N}} \arrow[d, two heads] \\ S \arrow[r, "A"'] & \mathsf{Closed} \end{tikzcd}$$

  Define $B(x) \leftrightarrow \forall_{n:\mathbb{N}}\beta_{x,n} = 0$. As $(i) \to (iii)$ by the above, B is the image of some Stone space. Note that A is the image of B, thus A is the image of some Stone space. □

Corollary 3.3.2 Closed subtypes of Stone spaces are Stone.

Corollary 3.3.3 For $S : \mathsf{Stone}$ and $A \subseteq S$ closed, we have $\exists_{x:S}A(x)$ is closed.

Proof By Corollary 3.3.2, $\Sigma_{x:S}A(x)$ is Stone, so its truncation is closed by Corollary 3.2.1. □

Corollary 3.3.4 Closed propositions are closed under sigma types.

Proof Let $P : \mathsf{Closed}$ and $Q : P \to \mathsf{Closed}$. Then $\Sigma_{p:P}Q(p) \leftrightarrow \exists_{p:P}Q(p)$. As P is Stone by Corollary 3.2.2, Corollary 3.3.3 gives that $\Sigma_{p:P}Q(p)$ is closed. □

Remark 3.3.5 Analogously to Corollary 2.2.3 and Remark 2.2.4, it follows that closedness is transitive and Closed forms a dominance.

Lemma 3.3.6 Assume $S : \mathsf{Stone}$ with $F, G : S \to \mathsf{Closed}$ be such that $F \cap G = \emptyset$. Then there exists a decidable subset $D : S \to 2$ such $F \subseteq D, G \subseteq \neg D$.

Proof Assume $S = Sp(B)$. By Theorem 3.3.1, for all $n : \mathbb{N}$ there is $f_n, g_n : B$ such that $x \in F$ if and only if $\forall_{n:\mathbb{N}}x(f_n) = 0$ and $y \in G$ if and only if $\forall_{n:\mathbb{N}}y(g_n) = 0$. Denote by h the sequence define by $h_{2k} = f_k$ and $h_{2k+1} = g_k$. Then $Sp(B/(h_k)_{k:\mathbb{N}}) = F \cap G = \emptyset$, so by Lemma 1.3.3 there exists finite sets $I, J \subseteq \mathbb{N}$ such that $1 =_B ((\bigvee_{i:I} f_i) \vee (\bigvee_{j:J} g_j))$. If $y \in F$, then $y(f_i) = 0$ for all $i : I$, hence $y(\bigvee_{j:J} g_j) = 1$ If $x \in G$, we have $x(\bigvee_{j:J} g_j) = 0$. Thus we can define the required D by $D(x) \leftrightarrow x(\bigvee_{j:J} g_j) = 1$. □

4 Compact Hausdorff spaces

Definition 4.0.1 A type X is called a compact Hausdorff space if its identity types are closed propositions and there exists some $S : \mathsf{Stone}$ and a surjection $S \twoheadrightarrow X$.

4.1 Topology on compact Hausdorff spaces

Lemma 4.1.1 Let $X : \mathsf{CHaus}$ with $S : \mathsf{Stone}$ and a surjective map $q : S \twoheadrightarrow X$. Then $A \subseteq X$ is closed if and only if it is the image of a closed subset of S by q.

Proof As q is surjective, we have $q(q^{-1}(A)) = A$. If A is closed, so is $q^{-1}(A)$ and hence A is the image of a closed subtype of S. Conversely, let $B \subseteq S$ be closed. Define $A' \subseteq S$ by

$$A'(s) = \exists_{t:S}(B(t) \wedge q(s) = q(t)).$$

Note that $B(t)$ and $q(s) = q(t)$ are closed. Hence by Corollary 3.3.3, A' is closed. Also A' factors through q as a map $A : X \to \mathsf{Closed}$. Furthermore, $A'(s) \leftrightarrow (q(s) \in q(B))$. Hence $A = q(B)$. □

Remark 4.1.2 Let $X : \mathsf{CHaus}$. From Theorem 3.3.1, it follows that $A \subseteq X$ is closed if and only if it is the image of a map $T \to X$ for some $T : \mathsf{Stone}$. If A is closed, it follows from Corollary 3.3.3 that $\exists_{x:X}A(x)$ is closed as well, hence $\neg\neg$-stable, and equivalent to $A \neq \emptyset$.

Corollary 4.1.3 For $U \subseteq X$ an open subset of a compact Hausdorff space, $\forall_{x:X}U(x)$ is open.

Lemma 4.1.4 Given $X : \mathsf{CHaus}$ and $C_n : X \to \mathsf{Closed}$ closed subsets such that $\bigcap_{n:\mathbb{N}} C_n = \emptyset$, there is some $k : \mathbb{N}$ with $\bigcap_{n\leq k} C_n = \emptyset$.

Proof By Lemma 4.1.1 it is enough to prove the result when X is Stone, and by Theorem 3.3.1 we can assume C_n decidable. So assume $X = Sp(B)$ and $c_n : B$ such that:

$$C_n = \{x : B \to 2 \mid x(c_n) = 0\}.$$

Then the set of maps $B \to 2$ sending all c_n to 0 is given by:

$$Sp(B/(c_n)_{n:\mathbb{N}}) \simeq \bigcap_{n:\mathbb{N}} C_n = \emptyset.$$

Hence $0 = 1$ in $B/(c_n)_{n:\mathbb{N}}$ and there is some $k : \mathbb{N}$ with $\bigvee_{n\leq k} c_n = 1$, which also means that:

$$\emptyset = Sp(B/(c_n)_{n\leq k}) \simeq \bigcap_{n\leq k} C_n$$

as required. □

Corollary 4.1.5 Let X, Y : CHaus and $f : X \to Y$. Suppose $(G_n)_{n:\mathbb{N}}$ is a decreasing sequence of closed subsets of X. Then $f(\bigcap_{n:\mathbb{N}} G_n) = \bigcap_{n:\mathbb{N}} f(G_n)$.

Proof It is always the case that $f(\bigcap_{n:\mathbb{N}} G_n) \subseteq \bigcap_{n:\mathbb{N}} f(G_n)$. For the converse direction, suppose that $y \in f(G_n)$ for all $n : \mathbb{N}$. We define $F \subseteq X$ closed by $F = f^{-1}(y)$. Then for all $n : \mathbb{N}$ we have that $F \cap G_n$ is non-empty. By Lemma 4.1.4 this implies that $\bigcap_{n:\mathbb{N}}(F \cap G_n) \neq \emptyset$. By Remark 4.1.2, $\bigcap_{n:\mathbb{N}}(F \cap G_n)$ is merely inhabited. Thus $y \in f(\bigcap_{n:\mathbb{N}} G_n)$ as required. □

Corollary 4.1.6 Let $A \subseteq X$ be a subset of a compact Hausdorff space and $p : S \twoheadrightarrow X$ be a surjective map with S : Stone. Then A is closed (resp. open) if and only if there exists a sequence $(D_n)_{n:\mathbb{N}}$ of decidable subsets of S such that $A = \bigcap_{n:\mathbb{N}} p(D_n)$ (resp. $A = \bigcup_{n:\mathbb{N}} \neg p(D_n)$).

Proof The characterization of closed sets follows from characterization (ii) in Theorem 3.3.1, Lemma 4.1.1 and Corollary 4.1.5. For open sets we use Remark 1.5.2 and Lemma 1.5.5. □

Remark 4.1.7 For S : Stone, there is a surjection $\mathbb{N} \twoheadrightarrow 2^S$. It follows that for any X : CHaus there is a surjection from $\mathbb{N}$ to a basis of X. Classically this means that X is second countable.

Lemma 4.1.8 Assume X : CHaus and $A, B \subseteq X$ closed such that $A \cap B = \emptyset$. Then there exist $U, V \subseteq X$ open such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Proof Let $q : S \to X$ be a surjective map with S : Stone. As $q^{-1}(A)$ and $q^{-1}(B)$ are closed, by Lemma 3.3.6, there is some $D : S \to 2$ such that $q^{-1}(A) \subseteq D$ and $q^{-1}(B) \subseteq \neg D$. Note that $q(D)$ and $q(\neg D)$ are closed by Lemma 4.1.1. We define $U = \neg q(\neg D) \cap \neg B$ and $V = \neg q(D) \cap \neg A$. As $q^{-1}(A) \cap \neg D = \emptyset$, we have that $A \subseteq \neg q(\neg D)$. As $A \cap B = \emptyset$, we have that $A \subseteq \neg B$ so $A \subseteq U$. Similarly $B \subseteq V$. Then U and V are disjoint because $\neg q(D) \cap \neg q(\neg D) \subseteq \neg(q(D) \cup q(\neg D)) = \neg X = \emptyset$. □

4.2 Compact Hausdorff spaces are stable under sigma types

Lemma 4.2.1 A type X is Stone if and only if it is merely a closed subset of $2^{\mathbb{N}}$.

Proof By Remark 1.1.4, any B : Boole can be written as $2[\mathbb{N}]/(r_n)_{n:\mathbb{N}}$. By Remark 2.3.3, the quotient map induces an embedding $Sp(B) \hookrightarrow Sp(2[\mathbb{N}]) = 2^{\mathbb{N}}$, which is closed by Theorem 3.3.1. □

Lemma 4.2.2 Compact Hausdorff spaces are stable under Σ-types.

Proof Assume X : CHaus and $Y : X \to \mathsf{CHaus}$. By Corollary 3.3.4 we have that identity type in $\Sigma_{x:X} Y(x)$ are closed. By Lemma 4.2.1 we know that for any $x : X$ there merely exists a closed $C \subseteq 2^{\mathbb{N}}$ with a surjection $\Sigma_{2^{\mathbb{N}}} C \twoheadrightarrow Y(x)$. By local choice we merely get S : Stone with a surjection $p : S \to X$ such that for all $s : S$ we have $C_s \subseteq 2^{\mathbb{N}}$ closed and a surjection $\Sigma_{2^{\mathbb{N}}} C_s \twoheadrightarrow Y(p(s))$. This gives a surjection $\Sigma_{s:S,x:2^{\mathbb{N}}} C_s(x) \twoheadrightarrow \Sigma_{x:X} Y_x$ and the source is Stone by Remark 3.1.4 and Corollary 3.3.2. □

4.3 Stone spaces are stable under sigma types

We will show that Stone spaces are precisely totally disconnected compact Hausdorff spaces. We will use this to prove that a sigma type of Stone spaces is Stone.

Lemma 4.3.1 Assume $X : \mathsf{CHaus}$, then 2^X is countably presented.

Proof There is some surjection $q : S \twoheadrightarrow X$ with $S : \mathsf{Stone}$. This induces an injection of Boolean algebras $2^X \hookrightarrow 2^S$. Note that $a : S \to 2$ lies in 2^X if and only if:

$$\forall_{s,t:S}\ q(s) =_X q(t) \to a(s) = a(t).$$

As equality in X is closed and equality in 2 is decidable, the implication is open for every $s, t : S$. By Corollary 4.1.3, we conclude that 2^X is an open subalgebra of 2^S. Therefore, it is in ODisc by Lemma 2.2.1 and Lemma 2.1.2 and in Boole by Corollary 2.3.2. □

Definition 4.3.2 For all $X : \mathsf{CHaus}$ and $x : X$, we define Q_x the connected component of x as the intersection of all $D \subseteq X$ decidable such that $x \in D$.

Lemma 4.3.3 For all $X : \mathsf{CHaus}$ with $x : X$, we have that Q_x is a countable intersection of decidable subsets of X.

Proof By Lemma 4.3.1, we can enumerate the elements of 2^X, say as $(D_n)_{n:\mathbb{N}}$. For $n : \mathbb{N}$ we define E_n as D_n if $x \in D_n$ and X otherwise. Then $\cap_{n:\mathbb{N}} E_n = Q_x$. □

Lemma 4.3.4 Assume $X : \mathsf{CHaus}$ with $x : X$ and suppose $U \subseteq X$ open with $Q_x \subseteq U$. Then we have some decidable $E \subseteq X$ with $x \in E$ and $E \subseteq U$.

Proof By Lemma 4.3.3, we have $Q_x = \bigcap_{n:\mathbb{N}} D_n$ with $D_n \subseteq X$ decidable. If $Q_x \subseteq U$, then

$$Q_x \cap \neg U = \bigcap_{n:\mathbb{N}} (D_n \cap \neg U) = \emptyset.$$

By Lemma 4.1.4 there is some $k : \mathbb{N}$ with

$$(\bigcap_{n \le k} D_n) \cap \neg U = \bigcap_{n \le k} (D_n \cap \neg U) = \emptyset.$$

Therefore $\bigcap_{n \le k} D_n \subseteq \neg\neg U$. As U is open, $\neg\neg U = U$ and $E := \bigcap_{n \le k} D_n$ is as desired. □

Lemma 4.3.5 Assume $X : \mathsf{CHaus}$ with $x : X$. Then any map in $Q_x \to 2$ is constant.

Proof Assume $Q_x = A \cup B$ with A, B decidable and disjoint subsets of Q_x. Assume $x \in A$. By Lemma 4.3.3, $Q_x \subseteq X$ is closed. Using Remark 3.3.5, it follows that $A, B \subseteq X$ are closed and disjoint. By Lemma 4.1.8 there exist U, V disjoint open such that $A \subseteq U$ and $B \subseteq V$. By Lemma 4.3.4 we have a decidable D such that $Q_x \subseteq D \subseteq U \cup V$. Note that $E := D \cap U = D \cap (\neg V)$ is clopen, hence decidable by Corollary 1.5.4. But $x \in E$, hence $B \subseteq Q_x \subseteq E$ but $B \cap E = \emptyset$, hence $B = \emptyset$. □

Lemma 4.3.6 Let $X : \mathsf{CHaus}$, then X is Stone if and only $\forall_{x:X}\ Q_x = \{x\}$.

Proof By Axiom 1, it is clear that for all $x : S$ with $S : \mathsf{Stone}$ we have that $Q_x = \{x\}$. Conversely, assume $X : \mathsf{CHaus}$ such that $\forall_{x:X}\ Q_x = \{x\}$. We claim that the evaluation map $e : X \to Sp(2^X)$ is both injective and surjective, hence an equivalence. Let $x, y : X$ be such that $e(x) = e(y)$, i.e. such that $f(x) = f(y)$ for all $f : 2^X$. Then $y \in Q_x$, hence $x = y$ by assumption. Thus e is injective. Let $q : S \twoheadrightarrow X$ be a surjective map. It induces an injection $2^X \hookrightarrow 2^S$, which by Axiom 2 induces a surjection $p : Sp(2^S) \twoheadrightarrow Sp(2^X)$. Note that $e \circ q$ is equal to p so e is surjective. □

Theorem 4.3.7
Assume $S : \mathsf{Stone}$ and $T : S \to \mathsf{Stone}$. Then $\Sigma_{x:S} T(x)$ is Stone.

Proof By Lemma 4.2.2 we have that $\Sigma_{x:S} T(x)$ is compact Hausdorff. By Lemma 4.3.6 it is enough to show that for all $x : S$ and $y : T(x)$ we have that $Q_{(x,y)}$ is a singleton. Assume $(x', y') \in Q_{(x,y)}$, then for any map $f : S \to 2$ we have that:

$$f(x) = f \circ \pi_1(x, y) = f \circ \pi_1(x', y') = f(x')$$

so that $x' \in Q_x$ and since S is Stone, by Lemma 4.3.6 we have that $x = x'$. Therefore we have $Q_{(x,y)} \subseteq \{x\} \times T(x)$. Assume $z, z' : Q_{(x,y)}$, then for any map $g : T(x) \to 2$ we have that $g(z) = g(z')$ by Lemma 4.3.5. Since $T(x)$ is Stone, we conclude $z = z'$ by Lemma 4.3.6. □

5 The unit interval as a Compact Hausdorff space

Since we have dependent choice, the unit interval $\mathbb{I} = [0, 1]$ can be defined using Cauchy reals or Dedekind reals. We can freely use results from constructive analysis [BB85]. As we have ¬WLPO, MP and LLPO, we can use the results from constructive reverse mathematics that follow from these principles [Ish06; Die18].

Definition 5.0.1 We define for each $n : \mathbb{N}$ the Stone space 2^n of binary sequences of length n. And we define $cs_n : 2^n \to \mathbb{Q}$ by $cs_n(\alpha) = \sum_{i<n} \frac{\alpha(i)}{2^{i+1}}$. Finally we write $\sim_n$ for the binary relation on 2^n given by $\alpha \sim_n \beta \leftrightarrow |cs_n(\alpha) - cs_n(\beta)| \leq \frac{1}{2^n}$.

Remark 5.0.2 The inclusion $Fin(n) \hookrightarrow \mathbb{N}$ induces a restriction $\_|_n : 2^{\mathbb{N}} \to 2^n$ for each $n : \mathbb{N}$.

Definition 5.0.3 We define $cs : 2^{\mathbb{N}} \to \mathbb{I}$ as $cs(\alpha) = \sum_{i:\mathbb{N}} \frac{\alpha(i)}{2^{i+1}}$.

Theorem 5.0.4
$\mathbb{I}$ is compact Hausdorff.

Proof By LLPO, we have that cs is surjective. Note that $cs(\alpha) = cs(\beta)$ if and only if for all $n : \mathbb{N}$ we have $\alpha|_n \sim_n \beta|_n$. This is a countable conjunction of decidable propositions. □

Remark 5.0.5 Following Definitions 2.7 and 2.10 of [BB85], we have that $x < y$ is open for all $x, y : \mathbb{I}$. Hence open intervals are open.

Lemma 5.0.6 For $D \subseteq 2^{\mathbb{N}}$ decidable, we have $cs(D)$ a finite union of closed intervals.

Proof If D is given by those $\alpha : 2^{\mathbb{N}}$ with a fixed initial segment, $cs(D)$ is a closed interval. Any decidable subset of $2^{\mathbb{N}}$ is a finite union of such subsets. □

Lemma 5.0.7 The complement of a finite union of closed intervals is a finite union of open intervals.

By Corollary 4.1.6 we can thus conclude:

Lemma 5.0.8 Every open $U \subseteq \mathbb{I}$ can be written as a countable union of open intervals.

It follows that the topology of $\mathbb{I}$ is generated by open intervals, which corresponds to the standard topology on $\mathbb{I}$. Hence our notion of continuity agrees with the ϵ, δ-definition of continuity one would expect and we get the following:

Theorem 5.0.9
Every function $f : \mathbb{I} \to \mathbb{I}$ is continuous in the ϵ, δ-sense.

6 Cohomology

In this section we compute $H^1(S, \mathbb{Z}) = 0$ for S Stone, and show that $H^1(X, \mathbb{Z})$ for X compact Hausdorff can be computed using Čech cohomology. We then apply this to compute $H^1(\mathbb{I}, \mathbb{Z}) = 0$.

Remark 6.0.1 We only work with the first cohomology group with coefficients in $\mathbb{Z}$ as it is sufficient for the proof of Brouwer's fixed-point theorem, but the results could be extended to $H^n(X, A)$ for A any family of countably presented abelian groups indexed by X.

Remark 6.0.2 We write Ab for the type of Abelian groups and if $G : Ab$ we write $\mathrm{B}G$ for the delooping of G [Pro13; Wär23]. This means that $H^1(X, G)$ is the set truncation of $X \to \mathrm{B}G$.

6.1 Čech cohomology

Definition 6.1.1 Given a type S, types T_x for $x : S$ and $A : S \to \mathrm{Ab}$, we define $\check{C}(S, T, A)$ as the chain complex:

$$\prod_{x:S} A_x^{T_x} \xrightarrow{d_0} \prod_{x:S} A_x^{T_x^2} \xrightarrow{d_1} \prod_{x:S} A_x^{T_x^3}$$

with the usual boundary maps:

$$d_0(\alpha)_x(u, v) = \alpha_x(v) - \alpha_x(u)$$
$$d_1(\beta)_x(u, v, w) = \beta_x(v, w) - \beta_x(u, w) + \beta_x(u, w)$$

Definition 6.1.2 Given a type S, types T_x for $x : S$ and $A : S \to \mathrm{Ab}$, we define its Čech cohomology groups by:

$$\check{H}^0(S,T,A) = \ker(d_0) \qquad \check{H}^1(S,T,A) = \ker(d_1)/\mathrm{im}(d_0)$$

We call elements of $\ker(d_1)$ cocycles and elements of $\mathrm{im}(d_0)$ coboundaries.

This means that $\check{H}^1(S,T,A) = 0$ if and only if $\check{C}(S,T,A)$ is exact. Now we give three general lemmas about Čech complexes.

Lemma 6.1.3 Assume a type S, types T_x for $x : S$ and $A : S \to \mathrm{Ab}$ with $t : \prod_{x:S} T_x$. Then $\check{C}(S,T,A)$ is exact.

Proof Assume given a cocycle, i.e. $\beta : \prod_{x:S} A_x^{T_x^2}$ such that for all $x : S$ and $u,v,w : T_x$ we have that $\beta_x(u,v) + \beta_x(v,w) = \beta_x(u,w)$. We define $\alpha : \prod_{x:S} A_x^{T_x}$ by $\alpha_x(u) = \beta_x(t_x,u)$. Then for all $x : S$ and $u,v : T_x$ we have that $d_0(\alpha)_x(u,v) = \beta_x(t_x,v) - \beta_x(t_x,u) = \beta_x(u,v)$ so that β is a coboundary. □

Lemma 6.1.4 Given a type S, types T_x for $x : S$ and $A : S \to \mathrm{Ab}$, we have that $\check{C}(S,T,\lambda x.A_x^{T_x})$ is exact.

Proof Assume given a cocycle, i.e. $\beta : \prod_{x:S} A_x^{T_x^3}$ such that for all $x : S$ and $u,v,w,t : T_x$ we have that $\beta_x(u,v,t) + \beta_x(v,w,t) = \beta_x(u,w,t)$. We define $\alpha : \prod_{x:S} A_x^{T_x^2}$ by $\alpha_x(u,t) = \beta_x(t,u,t)$. Then for all $x : S$ and $u,v,t : T_x$ we have that $d_0(\alpha)_x(u,v,t) = \beta_x(t,v,t) - \beta_x(t,u,t) = \beta_x(u,v,t)$ so that β is a coboundary. □

Lemma 6.1.5 Assume a type S and types T_x for $x : S$ such that $\prod_{x:S} \|T_x\|$ and $A : S \to \mathrm{Ab}$ such that $\check{C}(S,T,A)$ is exact. Then given $\alpha : \prod_{x:S} \mathrm{B}A_x$ with $\beta : \prod_{x:S} (\alpha(x) = *)^{T_x}$, we can conclude $\alpha = *$.

Proof We define $g : \prod_{x:S} A_x^{T_x^2}$ by $g_x(u,v) = \beta_x(u)^{-1} \cdot \beta_x(v)$. It is a cocycle in the Čech complex, so that by exactness there is $f : \prod_{x:S} A_x^{T_x}$ such that for all $x : S$ and $u,v : T_x$ we have that $g_x(u,v) = f_x(u)^{-1} \cdot f_x(v)$. Then we define $\beta' : \prod_{x:S} (\alpha(x) = *)^{T_x}$ by $\beta'_x(u) = \beta_x(u) \cdot f_x(u)^{-1}$ so that for all $x : S$ and $u,v : T_x$ we have that $\beta'_x(u) = \beta'_x(v)$ is equivalent to $f_x(u)^{-1} \cdot f_x(v) = \beta_x(u)^{-1} \cdot \beta_x(v)$, which holds by definition. Therefore β' factors through S, giving a proof of $\alpha = *$. □

6.2 Cohomology of Stone spaces

Lemma 6.2.1 Assume given $S : \mathsf{Stone}$ and $T : S \to \mathsf{Stone}$ such that $\prod_{x:S} \|T(x)\|$. Then there exists a sequence of finite types $(S_k)_{k:\mathbb{N}}$ with limit S and a compatible sequence of families of finite types T_k over S_k with $\prod_{x:S_k} \|T_k(x)\|$ and $\lim_k \left(\sum_{x:S_k} T_k(x)\right) = \sum_{x:S} T(x)$.

Proof This follows from Remark 3.1.4 and Theorem 4.3.7. □

Lemma 6.2.2 Assume given $S : \mathsf{Stone}$ with $T : S \to \mathsf{Stone}$ such that $\prod_{x:S} \|T_x\|$. Then we have that $\check{C}(S,T,\mathbb{Z})$ is exact.

Proof We apply lemma 6.2.1 to get S_k and T_k finite. Then by corollary 3.1.7 we have that $\check{C}(S,T,\mathbb{Z})$ is the sequential colimit of the $\check{C}(S_k,T_k,\mathbb{Z})$. By lemma 6.1.3 we have that each of the $\check{C}(S_k,T_k,\mathbb{Z})$ is exact, and a sequential colimit of exact sequences is exact. □

Lemma 6.2.3 Given $S : \mathsf{Stone}$, we have that $H^1(S,\mathbb{Z}) = 0$.

Proof Assume given a map $\alpha : S \to \mathrm{B}\mathbb{Z}$. We use local choice to get $T : S \to \mathsf{Stone}$ such that $\prod_{x:S} \|T_x\|$ with $\beta : \prod_{x:S} (\alpha(x) = *)^{T_x}$. Then we conclude by lemma 6.2.2 and lemma 6.1.5. □

Corollary 6.2.4 For any $S : \mathsf{Stone}$ the canonical map $\mathrm{B}(\mathbb{Z}^S) \to (\mathrm{B}\mathbb{Z})^S$ is an equivalence.

6.3 Čech cohomology of compact Hausdorff spaces

Definition 6.3.1 A Čech cover consists of X : CHaus and $S : X \to$ Stone such that $\prod_{x:X} \|S_x\|$ and $\sum_{x:X} S_x$: Stone.

By definition any compact Hausdorff type has a Čech cover.

Lemma 6.3.2 Given a Čech cover (X, S), we have that $H^0(X, \mathbb{Z}) = \check{H}^0(X, S, \mathbb{Z})$.

Proof By definition an element in $\check{H}^0(X, S, \mathbb{Z})$ is a map $f : \prod_{x:X} \mathbb{Z}^{S_x}$ such that for all $u, v : S_x$ we have $f(u) = f(v)$. Since $\mathbb{Z}$ is a set and the S_x are merely inhabited, this is equivalent to $\mathbb{Z}^X$. □

Lemma 6.3.3 Given a Čech cover (X, S) we have an exact sequence:

$$H^0(X, \lambda x.\mathbb{Z}^{S_x}) \to H^0(X, \lambda x.\mathbb{Z}^{S_x}/\mathbb{Z}) \to H^1(X, \mathbb{Z}) \to 0$$

Proof We use the long exact cohomology sequence associated to:

$$0 \to \mathbb{Z} \to \mathbb{Z}^{S_x} \to \mathbb{Z}^{S_x}/\mathbb{Z} \to 0$$

We just need $H^1(X, \lambda x.\mathbb{Z}^{S_x}) = 0$ to conclude. But by corollary 6.2.4 we have that $H^1(X, \lambda x.\mathbb{Z}^{S_x}) = H^1\left(\sum_{x:X} S_x, \mathbb{Z}\right)$ which vanishes by lemma 6.2.3. □

Lemma 6.3.4 Given a Čech cover (X, S) we have an exact sequence:

$$\check{H}^0(X, \lambda x.\mathbb{Z}^{S_x}) \to \check{H}^0(X, \lambda x.\mathbb{Z}^{S_x}/\mathbb{Z}) \to \check{H}^1(X, \mathbb{Z}) \to 0$$

Proof By lemma 6.2.3 and the long exact sequence for cohomology, we have an exact sequence of complexes:

$$0 \to \check{C}(X, S, \mathbb{Z}) \to \check{C}(X, S, \lambda x.\mathbb{Z}^{S_x}) \to \check{C}(X, S, \lambda x.\mathbb{Z}^{S_x}/\mathbb{Z}) \to 0$$

But since $\check{H}^1(X, \lambda x.\mathbb{Z}^{S_x}) = 0$ by lemma 6.1.4, we conclude using the associated long exact sequence. □

Theorem 6.3.5
Given a Čech cover (X, S), we have that $H^1(X, \mathbb{Z}) = \check{H}^1(X, S, \mathbb{Z})$

Proof We apply lemma 6.3.2, lemma 6.3.3 and lemma 6.3.4. □

This means that Čech cohomology does not depend on S.

6.4 Cohomology of the interval

Remark 6.4.1 Recall from Definition 5.0.1 that there is a binary relation $\sim_n$ on $2^n =: \mathbb{I}_n$ such that $(2^n, \sim_n)$ is equivalent to $(\mathrm{Fin}(2^n), \lambda x, y.\ |x - y| \leq 1)$ and for $\alpha, \beta : 2^{\mathbb{N}}$ we have $(cs(\alpha) = cs(\beta)) \leftrightarrow (\forall_{n:\mathbb{N}} \alpha|_n \sim_n \beta|_n)$.

We define $\mathbb{I}_n^{\sim 2} = \Sigma_{x,y:\mathbb{I}_n} x \sim_n y$ and $\mathbb{I}_n^{\sim 3} = \Sigma_{x,y,z:\mathbb{I}_n} x \sim_n y \wedge y \sim_n z \wedge x \sim_n z$.

Lemma 6.4.2 For any $n : \mathbb{N}$ we have an exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z}^{\mathbb{I}_n} \to \mathbb{Z}^{\mathbb{I}_n^{\sim 2}} \to \mathbb{Z}^{\mathbb{I}_n^{\sim 3}}$$

with the obvious boundary maps.

Proof It is clear that the map $\mathbb{Z} \to \mathbb{Z}^{\mathbb{I}_n}$ is injective as $\mathbb{I}_n$ is inhabited, so the sequence is exact at $\mathbb{Z}$. Assume a cocycle $\alpha : \mathbb{Z}^{\mathbb{I}_n}$, meaning that for all $u, v : \mathbb{I}_n$, if $u \sim_n v$ then $\alpha(u) = \alpha(v)$. Then by remark 6.4.1 we see that α is constant, so the sequence is exact at $\mathbb{Z}^{\mathbb{I}_n}$.

Assume a cocycle $\beta : \mathbb{Z}^{\mathbb{I}_n^{\sim 2}}$, meaning that for all $u, v, w : \mathbb{I}_n$ such that $u \sim_n v$, $v \sim_n w$ and $u \sim_n w$ we have that $\beta(u, v) + \beta(v, w) = \beta(u, w)$. Using remark 6.4.1 we can define $\alpha(n) = \beta(0, 1) + \cdots + \beta(n-1, n)$. We can check that $\beta(m, n) = \alpha(n) - \alpha(m)$, so β is indeed a coboundary and the sequence is exact at $\mathbb{Z}^{\mathbb{I}_n^{\sim 2}}$. □

Proposition 6.4.3 We have that $H^0(\mathbb{I}, \mathbb{Z}) = \mathbb{Z}$ and $H^1(\mathbb{I}, \mathbb{Z}) = 0$.

Proof Consider $cs : 2^{\mathbb{N}} \to \mathbb{I}$ and the associated Čech cover T of $\mathbb{I}$ defined by:

$$T_x = \Sigma_{y:2^{\mathbb{N}}}(x =_{\mathbb{I}} cs(y))$$

Then for $l = 2, 3$ we have that $\lim_n \mathbb{I}_n^{\sim l} = \sum_{x:\mathbb{I}} T_x^l$. By lemma 6.4.2 and stability of exactness under sequential colimit, we have an exact sequence:

$$0 \to \mathbb{Z} \to \mathrm{colim}_n \mathbb{Z}^{\mathbb{I}_n} \to \mathrm{colim}_n \mathbb{Z}^{\mathbb{I}_n^{\sim 2}} \to \mathrm{colim}_n \mathbb{Z}^{\mathbb{I}_n^{\sim 3}}$$

By corollary 3.1.7 this sequence is equivalent to:

$$0 \to \mathbb{Z} \to \Pi_{x:\mathbb{I}} \mathbb{Z}^{T_x} \to \Pi_{x:\mathbb{I}} \mathbb{Z}^{T_x^2} \to \Pi_{x:\mathbb{I}} \mathbb{Z}^{T_x^3}$$

So it being exact implies that $\check{H}^0(\mathbb{I}, T, \mathbb{Z}) = \mathbb{Z}$ and $\check{H}^1(\mathbb{I}, T, \mathbb{Z}) = 0$. We conclude by lemma 6.3.2 and theorem 6.3.5. ☐

Remark 6.4.4 We could carry a similar computation for $\mathbb{S}^1$, by approximating it with 2^n with $0^n \sim_n 1^n$ added. We would find $H^1(\mathbb{S}^1, \mathbb{Z}) = \mathbb{Z}$.

6.5 Brouwer's fixed-point theorem

Here we consider the modality defined by localising at $\mathbb{I}$ [RSS20], denoted by $L_{\mathbb{I}}$. We say that X is $\mathbb{I}$-local if $L_{\mathbb{I}}(X) = X$ and that it is $\mathbb{I}$-contractible if $L_{\mathbb{I}}(X) = 1$.

Lemma 6.5.1 $\mathbb{Z}$ and 2 are $\mathbb{I}$-local.

Proof By proposition 6.4.3, from $H^0(\mathbb{I}, \mathbb{Z}) = \mathbb{Z}$ we get that the map $\mathbb{Z} \to \mathbb{Z}^{\mathbb{I}}$ is an equivalence, so $\mathbb{Z}$ is $\mathbb{I}$-local. We see that 2 is $\mathbb{I}$-local as it is a retract of $\mathbb{Z}$. ☐

Remark 6.5.2 Since 2 is $\mathbb{I}$-local, we know by duality that any Stone space is $\mathbb{I}$-local.

Lemma 6.5.3 $\mathrm{B}\mathbb{Z}$ is $\mathbb{I}$-local.

Proof Any identity type in $\mathrm{B}\mathbb{Z}$ is a $\mathbb{Z}$-torsor, so it is $\mathbb{I}$-local by lemma 6.5.1. So there is at most one factorisation of any map $\mathbb{I} \to \mathrm{B}\mathbb{Z}$ through 1. From $H^1(\mathbb{I}, \mathbb{Z}) = 0$ we get that there merely exists such a factorisation. ☐

Lemma 6.5.4 Assume X a type with $x : X$ such that for all $y : X$ we have $f : \mathbb{I} \to X$ such that $f(0) = x$ and $f(1) = y$. Then X is $\mathbb{I}$-contractible.

Proof For all $y : X$ we get a map $g : \mathbb{I} \to X \to L_{\mathbb{I}}(X)$ such that $g(0) = [x]$ and $g(1) = [y]$. Since $L_{\mathbb{I}}(X)$ is $\mathbb{I}$-local this means that $\prod_{x:X}[*] = [x]$. We conclude $\prod_{y:L_{\mathbb{I}}(X)}[x] = y$ by applying the elimination principle for the modality. ☐

Corollary 6.5.5 We have that $\mathbb{R}$ and $\mathbb{D}^2 = \{x, y : \mathbb{R} \mid x^2 + y^2 \leq 1\}$ are $\mathbb{I}$-contractible.

Proposition 6.5.6 $L_{\mathbb{I}}(\mathbb{R}/\mathbb{Z}) = \mathrm{B}\mathbb{Z}$.

Proof As for any group quotient, the fibers of the map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ are $\mathbb{Z}$-torsors, se we have an induced pullback square:

$$\begin{tikzcd} \mathbb{R} \arrow[r] \arrow[d] & 1 \arrow[d] \\ \mathbb{R}/\mathbb{Z} \arrow[r] & \mathrm{B}\mathbb{Z} \end{tikzcd}$$

Now we check that the bottom map is an $\mathbb{I}$-localisation. Since $\mathrm{B}\mathbb{Z}$ is $\mathbb{I}$-local by lemma 6.5.3, it is enough to check that its fibers are $\mathbb{I}$-contractible. Since $\mathrm{B}\mathbb{Z}$ is connected it is enough to check that $\mathbb{R}$ is $\mathbb{I}$-contractible, but this is corollary 6.5.5. ☐

Remark 6.5.7 By lemma 6.5.3, for any X we have that $H^1(X, \mathbb{Z}) = H^1(L_{\mathbb{I}}(X), \mathbb{Z})$, so that by proposition 6.5.6 we have that $H^1(\mathbb{R}/\mathbb{Z}, \mathbb{Z}) = H^1(\mathrm{B}\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$.

We omit the proof that $\mathbb{S}^1 = \{x, y : \mathbb{R} \mid x^2 + y^2 = 1\}$ is equivalent to $\mathbb{R}/\mathbb{Z}$. The equivalence can be constructed using trigonometric functions, which exists by Proposition 4.12 in [BB85].

Proposition 6.5.8 The map $\mathbb{S}^1 \to \mathbb{D}^2$ has no retraction.

Proof By corollary 6.5.5 and proposition 6.5.6 we would get a retraction of $\mathrm{B}\mathbb{Z} \to 1$, so $\mathrm{B}\mathbb{Z}$ would be contractible. □

Theorem 6.5.9 (Intermediate value theorem)
For any $f : \mathbb{I} \to \mathbb{I}$ and $y : \mathbb{I}$ such that $f(0) \leq y$ and $y \leq f(1)$, there exists $x : \mathbb{I}$ such that $f(x) = y$.

Proof By Remark 4.1.2, the proposition $\exists x : \mathbb{I}. f(x) = y$ is closed and therefore $\neg\neg$-stable, so we can proceed with a proof by contradiction. If there is no such $x : \mathbb{I}$, we have $f(x) \neq y$ for all $x : \mathbb{I}$. It is a standard fact of constructive analysis [BB85], that for different numbers $a, b : \mathbb{I}$, we have $a < b$ or $b < a$, so the following two sets cover $\mathbb{I}$:

$$U_0 := \{x : \mathbb{I} \mid f(x) < y\} \qquad U_1 := \{x : \mathbb{I} \mid y < f(x)\}$$

Since U_0 and U_1 are disjoint, we have $\mathbb{I} = U_0 + U_1$ which allows us to define a non-constant function $\mathbb{I} \to 2$, which contradicts Lemma 6.5.1. □

Theorem 6.5.10 (Brouwer's fixed-point theorem)
For all $f : \mathbb{D}^2 \to \mathbb{D}^2$ there exists $x : \mathbb{D}^2$ such that $f(x) = x$.

Proof As above, by Remark 4.1.2, we can proceed with a proof by contradiction, so we assume $f(x) \neq x$ for all $x : \mathbb{D}^2$. For any $x : \mathbb{D}^2$ we set $d_x = x - f(x)$, so we have that one of the coordinates of d_x is invertible. Let $H_x(t) = f(x) + t \cdot d_x$ be the line through x and $f(x)$. The intersections of H_x and $\partial\mathbb{D}^2 = \mathbb{S}^1$ are given by the solutions of an equation quadratic in t. By invertibility of one of the coordinates of d_x, there is exactly one solution with $t > 0$. We denote this intersection by $r(x)$ and the resulting map $r : \mathbb{D}^2 \to \mathbb{S}^1$ has the property that it preserves $\mathbb{S}^1$. Then r is a retraction from $\mathbb{D}^2$ onto its boundary $\mathbb{S}^1$, which is a contradiction by Proposition 6.5.8. □

7 Urysohn

For the purposes of this exercise, we let a dyadic number of length n be $cs_n(d)$ for a finite sequence $d : 2^n$. If the number corresponds to $\frac{1}{2^n}$ or $\frac{2^n-1}{2^n}$, we call it an outer dyadic number, otherwise we call it an inner dyadic number.

Lemma 7.0.1 Denote $q : 2^{\mathbb{N}} \to \mathbb{I}$. Given a function $a : \mathbb{N} \to 2^{\mathbb{N}}$ such that $|cs_n(a|_n) - cs_{n+1}(a|_{n+1})| \leq \frac{1}{2^n+1}$ the limit of $q(a_n) : \mathbb{I}$ exists as number in $\mathbb{I}$.

Proof Define $a' : 2^{\mathbb{N}}$ inductively by $a'(n) = \begin{cases} 1 \text{ if } cs_{n+1}(a|_{n+1}) > cs_{n-1}(a'|_{n-1}) + \frac{1}{2^n} \\ 0 \text{ otherwise} \end{cases}$ Then a converges to $cs(a')$. □

Lemma 7.0.2 Let $S : \mathsf{Stone}$ and let $A, B \subseteq S$ closed and disjoint. Then there merely exists a function $f : S \to \mathbb{I}$ such that $f(a) = 0$ if $a \in A$ and $f(b) = 1$ if $b \in B$.

Proof We will use dependent choice to define for each $n : \mathbb{N}$ and for each dyadic number d of length n with $d \neq 0, 1$, a decidable subset $D_d \subseteq S$ satisfying $A \subseteq D_d \subseteq \neg B$. And whenever $d \leq d'$ we will have $D_d \subseteq D_{d'}$.

By Lemma 3.3.6, we can find $D(\frac{1}{2})$ with $A \subseteq D \subseteq \neg B$. Assume given D_d for all dyadic $d \neq 0, 1$ of length n. To show that there exist D_d for all new dyadic $d \neq 0, 1$ of length $n+1$ we make a case distinction on whether d is outer or inner.

- If d is inner, we have that $D_{d \pm \frac{1}{2^{n+1}}}$ are already defined, and we apply Lemma 3.3.6 to find D_d.
- If $d = \frac{1}{2^{n+1}}$, we apply Lemma 3.3.6 to $D_{\frac{1}{2^n}}$ and A.
- If $d = \frac{2^{n+1}-1}{2^{n+1}}$, we apply Lemma 3.3.6 to $D_{\frac{2^n-1}{2^n}}$ and B.

Using countable choice, we end up with an $2^{\mathbb{N}}$-indexed tree of decidable subsets of S as required.

We then define $f : \Pi_{n:\mathbb{N}}(S \to 2^n)$ by $f(n, s)$ the lowest dyadic number d of length n with $D_d(s)$ if it exists, and 1 otherwise. Note that $f(n, s) \geq f(n+1, s) \geq f(n, s) - \frac{1}{2^{n+1}}$ for all $n : \mathbb{N}$. Therefore for each $s : S$ the sequence $f(n, s)$ converges to a real number in $\mathbb{I}$, and thus f defines a function $S \to \mathbb{I}$.

By design for every $a \in A$ we have $a \in D_d$ for all d, hence $f(a) = 0$ and for every $b \in B$ we have $b \notin D_d$ for all d hence $f(b) = 1$. □

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