Synthetic Stone Duality: Summary

Felix Cherubini, Thierry Coquand, Freek Geerligs and Hugo Moeneclaey

December 3, 2024

We give a very quick overview of Synthetic Stone Duality.

1 Axioms

Definition 1.1 For B a countably presented Boolean algebra, we define $Sp(B)$ as the set of Boolean morphisms from B to 2.

Definition 1.2 Given a type X, we say that it is Stone if there exists a countably presented boolean algebra B such that:

$$X = Sp(B)$$

Axiom 1 (Stone duality)
For any countably presented Boolean algebra B, the evaluation map $B \to 2^{Sp(B)}$ is an isomorphism.

Axiom 2 (Surjections are formal Surjections)
A map $f : Sp(B') \to Sp(B)$ is surjective if and only if the corresponding map $B \to B'$ is injective.

Axiom 3 (Local choice)
Whenever S Stone and $E \twoheadrightarrow S$ surjective, then there exists some T Stone, a surjection $T \twoheadrightarrow S$ and a map $T \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[d, two heads] & \\ S & T \arrow[l, two heads] \arrow[ul] \end{tikzcd} \tag{1}$$

Axiom 4 (Dependent choice)
Given a sequence of surjections:

$$X_0 \twoheadleftarrow X_1 \twoheadleftarrow X_2 \twoheadleftarrow \cdots$$

we have an induced surjection:

$$X_0 \twoheadleftarrow \varprojlim X_k$$

We define the type Boole of countably presented Boolean algebras, the type Stone of Stone spaces.

A type X is Compact Hausdorff iff we can find $S : \mathsf{Stone}$ with a surjective map $S \to X$ such that the kernel $S \times_X S$ is Stone. We write CHaus the type of Compact Hausforff spaces.

2 Omniscience Principles

A proposition is *open* iff it is of the form $\exists_n a_n$ with a_n decidable. It is *closed* if it is of the form $\forall_n a_n$ with a_n decidable. We write Open (resp. Closed) the set of open (resp. closed) propositions.

Theorem 2.1 (The negation of the weak limited principle of omniscience)
It is not the case that for all $\alpha : \mathbb{N} \to 2$ we can decide whether $\forall (n : \mathbb{N}).\ \alpha(n) = 0$.

Theorem 2.2 (Markov's principle)
For all $\alpha : \mathbb{N} \to 2$, we have that:

$$\neg(\forall (n : \mathbb{N}).\ \alpha(n) = 0) \to \exists (n : \mathbb{N}).\ \alpha(n) = 1$$

This can be rephrased as: the negation of a closed proposition is open. It is direct that the negation of an open proposition is closed.

It follows that both open and closed propositions are not not stable. (They are not decidable in general.)

Definition 2.3 Let $\mathbb{N}_\infty$ be the type of sequence:

$$\alpha : \mathbb{N} \to 2$$

where α has value 1 at most one. We have $\mathbb{N}_\infty : \mathsf{Stone}$.

Theorem 2.4 (The lesser limited principle of omniscience (LLPO))
For $\alpha : \mathbb{N}_\infty$, we have that

$$(\forall (k : \mathbb{N}).\ \alpha(2k) = 0) \vee (\forall (k : \mathbb{N}).\ \alpha(2k+1) = 0) \tag{2}$$

This can be rephrased as the fact that the map $\mathbb{N}_\infty + \mathbb{N}_\infty \to \mathbb{N}_\infty$ sending $\mathsf{inl}(\alpha)$ to $\lambda_k \alpha(2k)$ and $\mathsf{inr}(\alpha)$ to $\lambda_k \alpha(2k+1)$ is *surjective.*

Since this map has no section, this shows that $\mathbb{N}_\infty$ is *not* projective. (David Wärn has noticed that $\mathbb{Z}[\mathbb{N}_\infty]$ is *not* internally projective in the category of Abelian groups.)

Yet another formulation of LLPO is that the disjunction of two closed propositions is closed. It is also equivalent to Brouwer's fixed point theorem and to Weak König's Lemma.

Still another formulation of LLPO is completness of propositional logic. Another way to state Axiom 2 is that $\neg\neg S \to \|S\|$ for $S : \mathsf{Stone}$.

3 Topology

Theorem 3.1
If B is a Boolean algebra we have $B : \mathsf{Boole}$ iff $B = \varinjlim B_n$ with B_n finite Boolean algebra. We have $S : \mathsf{Stone}$ iff $S = \varprojlim S_n$ with S_n finite sets.

Theorem 3.2
If $S : \mathsf{Stone}$, a subset $F : \mathsf{Stone}$ is a countable intersection of decidable subsets iff it is classified by Closed. It is a countable disjunction of decidable subsets iff it is classified by Open. If $S : \mathsf{Stone}$ and $x_0, x_1 : S$ then $x_0 =_S x_1$ is closed. If P is a proposition then $P : \mathsf{Closed}$ iff $P : \mathsf{Stone}$ iff $P : \mathsf{CHaus}$.

Theorem 3.3
If $X : \mathsf{CHaus}$ with a surjective map $S \to X$ with $S \times_X S : \mathsf{Stone}$, a subset of X is classified by a Closed iff it is the image of a closed subset of S.

Definition 3.4 The unit interval $[0,1]$ is the image of Cantor space by the map $\alpha \mapsto \Sigma_n \alpha(n)/2^n$.

Any map $f : Y \to X$ is continuous in the sense that the inverse image of an open subset of X is an open subset of Y.

Theorem 3.5
If $S : \mathsf{Stone}$ two disjoint closed subsets of S are separated by a decidable subset. If $X : \mathsf{CHaus}$ and A, B are two disjoint closed subsets of X, then there exists disjoint open subsets U, V of X such that $A \subseteq U$ and $B \subseteq V$.

Theorem 3.6
If $X : \mathsf{CHaus}$ and F_n is a decreasing sequence of closed subsets of X such that $\cap_n F_n = \emptyset$ then there exists n such that $F_n = \emptyset$. If U_n is an increasing sequence of open subsets of X such that $X = \cup_n U_n$ then there exists n such that $X = U_n$.

4 Directed Univalence

Based on Barton-Commelin axioms.

Definition 4.1 A type is overtly discrete if it merely is a sequential colimit of finite type.

We write ODisc for the type of overtly discrete sets.

Proposition 4.2 If $E : \mathsf{ODisc}$ and $e_0, e_1 : E$ then $e_0 =_E e_1$ is open. If P is a proposition, then $P : \mathsf{ODisc}$ iff $P : \mathsf{Open}$.

Proposition 4.3 If B is a Boolean algebra, we have $B : \mathsf{Boole}$ iff $B : \mathsf{ODisc}$.

Theorem 4.4
Given X, Y : ODisc, the fiber of the map:

$$(\text{Open} \to \text{ODisc}) \to \text{ODisc} \times \text{ODisc}$$

$$P \mapsto (P(\bot), P(\top))$$

over (X, Y) is:

$$X \to Y$$

Intuitively this means that the type of open propositions is a directed interval for overtly discrete types.

In particular, any map $\mathsf{Open} \to \mathsf{Open}$ is monotone.

Theorem 4.5 (Tychonoff)
If $E : \mathsf{ODisc}$ and $X : E \to \mathsf{CHaus}$ then $\Pi_E X : \mathsf{CHaus}$. If $X : \mathsf{CHaus}$ and $E : X \to \mathsf{ODisc}$ then $\Pi_X E : \mathsf{ODisc}$. In particular if $E : X \to \mathsf{Open}$ then $\Pi_X E : \mathsf{Open}$.

5 Cohomology

Theorem 5.1
Let S be a Stone space, then for all $i > 0$ we have

$$H^i(S, \mathbb{Z}) = 0$$

If $X : \mathsf{CHaus}$ and $S : \mathsf{Stone}$ with $S \to X$ surjective, of fiber S_x for $x : X$, we can consider the cochain complex

$$\Pi_{x:X}\mathbb{Z}^{S_x} \to \Pi_{x:X}\mathbb{Z}^{S_x \times S_x} \to \Pi_{x:X}\mathbb{Z}^{S_x \times S_x \times S_x} \to \dots$$

then the cohomology groups $H^n(X, \mathbb{Z})$, in the sense of univalent type theory, are exactly the cohomology groups of this cochain complex. In particular, we have the following.

Proposition 5.2 For all $i > 0$ we have

$$H^i([0, 1], \mathbb{Z}) = 0$$

Similar to real-cohesion, we can construct a shape modality, which we expect to map finite CW-complexes to their fundamental ∞-groupoids:

Definition 5.3 $\int$ is the modality given by nullification at the interval $[0, 1]$.

Proposition 5.4 The shape of the topological circle $\mathbb{S}^1 :\equiv \mathbb{R}/\mathbb{Z}$ is the higher inductive circle:

$$\int(\mathbb{S}^1) = \mathrm{K}(\mathbb{Z}, 1) = S^1$$

Since $K(\mathbb{Z}, n)$ is $\int$-modal, we also have:

$$H^n(\mathbb{S}^1, \mathbb{Z}) = H^n(\int \mathbb{S}^1, \mathbb{Z}) = H^n(S^1, \mathbb{Z})$$

So we can use computations in plain homotopy type theory to get:

$$H^1(\mathbb{S}^1, \mathbb{Z}) = \mathbb{Z}$$
$$H^n(\mathbb{S}^1, \mathbb{Z}) = 0 \text{ for } n > 1$$