Model for Synthetic Stone Duality

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Going toward a model for synthetic Stone duality, by Thierry Coquand, Jonas Höfer and Hugo Moeneclaey.

It is very drafty, in particular I (Hugo Moeneclaey) reuse many results from synthetic algebraic geometry freely, without checking they still hold for c.p. boolean algebras.

Contents

1 Presheaf model

In this section the goal is to build the presheaf model, which should satisfies the following:

- There is a boolean ring $\mathbb{B}$.
- For all c.p. boolean $\mathbb{B}$-algebra C, the map:

$$C \to \mathbb{B}^{\mathrm{Spec}(C)}$$

  is an equivalence.
- Affine scheme have choice.
- Dependent choice.

2 Sheafification

In this section we will give general consideration about sheafification and local choice, essentially we need the following:

Theorem 2.0.1
If T is a subcanonical toplogy, then the interpretation of the following holds in T-sheaves:

(i) For any $X \in T$ we have $\|X\|$.

(ii) Duality.

(iii) Any affine scheme has T-local choice.

And perhaps the same in the Zariski topos (where the topology has to include the Zariski topology):

Theorem 2.0.2
Let T be a subcanonical topology in the Zariski topos, then T-sheaves enjoys the following:

(i) For any $X \in T$ we have $\|X\|$.

(ii) Duality.

(iii) Any affine scheme has T-local choice.

I still don't know whether we really should go in two steps.

3 From presheaves to Zariski sheaves

We want to do two successive sheafifications, the Zariski one and the fppf one. We could (should?) do them in one go. This section is about the Zariski sheafification, meaning we build a model of:

- Duality for 2
- Affine schemes are projective.
- Dependent choice.

3.1 Zariski sheaves

Definition 3.1.1 The Zariski topology consists of finite sums:

$$f_1\text{inv} + \cdots + f_n\text{inv}$$

where $f_1, \cdots, f_n : B$ such that:

$$(f_1, \cdots, f_n) = 1$$

Definition 3.1.2 The disjoint topology consists of finite sums:

$$f_1\text{inv} + \cdots + f_n\text{inv}$$

where $f_1, \cdots, f_n : B$ is a fundamental system of idempotent, i.e.

$$f_1 + \cdots + f_n = 1$$
$$f_i^2 = f_i$$
$$f_i f_j = 0$$

when $i \neq j$.

Lemma 3.1.3 A type X is a Zariski sheaf if and only if it is a disjoint sheaf.

Proof The disjoint topology is included on the Zariski topology, so we just need to prove that:

$$f_1\text{inv} \vee \cdots \vee f_n\text{inv}$$

is disjoint-contractible. So we need to prove the proposition assuming $f = 0 \vee f = 1$ for finitely many $f : B$. This is straightforward. □

Lemma 3.1.4 The Zariski topology is subcanonical.

Proof We will prove later the stronger result that the fppf topology is subcanonical. □

Next proposition is based on theorem 2.0.1, which has not be proven in details yet.

Proposition 3.1.5 The Zariski topos satisfy the following:

- For all c.p. boolean ring C, the map:

$$C \to 2^{\text{Spec}(C)}$$

  is an equivalence.
- For all c.p. boolean ring C, the type $\text{Spec}(C)$ has choice.

Proof We apply theorem 2.0.1.

- Any local boolean ring is 2, so the generic ring in the Zariski topos is 2, justifying the first item.
- We have disjoint-local choice for affine schemes, let us check that it implies actual choice. To do this it is enough to check that disjoint-cover are equivalence, i.e. that given a fundamental system $f_1, \cdots, f_n$ of idempotent in 2 we have that:

$$f_1 = 1 + \cdots + f_n = 1$$

  is contractible. This is straightforward. □

3.2 Dependent choice from presheaves to Zariski sheaves

Lemma 3.2.1 Assume given a type X and $A_1, \cdots, A_n$ types such that for all $I \subset [1,n]$ with at least two elements, $X^{\prod_{i:I} A_i}$ is a proposition. Then the map:

$$A_1 + \cdots + A_n \to A_1 \vee \cdots \vee A_n$$

induces an equivalence:

$$X^{A_1 \vee \cdots \vee A_n} \to X^{A_1 + \cdots + A_n}$$

Proof Like Cech cohomology stuff, TODO □

Lemma 3.2.2 Assume given $f_1, \cdots, f_n : \mathbb{B}$ a fundamental system of idempotents and X a Zariski sheaf. Then X is $(f_1 = 1 + \cdots + f_n = 1)$-local.

Proof For all $i \neq j$ we have that $f_i = 1 \wedge f_j = 1$ is equivalent to $0 = 1$ as $f_i f_j = 0$, and so is the conjunction of any two or more $f_i = 1$.

But we have that $X^{0=1}$ is contractible when X is a Zariski sheaf, as $0 = 1$ implies any Zariski sheaf contractible.

So we can apply lemma 3.2.1 to conclude that since X is $f_1 = 1 \vee \cdots \vee f_n = 1$-local it is indeed $(f_1 = 1 + \cdots + f_n = 1)$-local. □

Remark 3.2.3 Alternatively we can prove that given $f_1, \cdots, f_n : \mathbb{B}$ a fundamental system of idempotents, the map:

$$f_1 = 1 + \cdots + f_n = 1 \to f_1 = 1 \vee \cdots \vee f_n = 1$$

has Zariski contractible fibres, indeed its fibers are of the form:

$$\top + 0 = 1 + \cdots + 0 = 1$$

which is Zariski equivalent to $\top$ as $0 = 1$ is Zariski equivalent to $\bot$. From this we conclude that Zariski sheaves are $(f_1 = 1 + \cdots + f_n = 1)$-local

Lemma 3.2.4 Assume given a Zariski sheaf X. Then $\|X\|$ is a Zariski sheaf.

Proof Assume $f_1, \cdots, f_n : \mathbb{B}$ a fundamental system of idempotent such that:

$$f_1 = 1 \vee \cdots \vee f_n = 1 \to \|X\|$$

We need to prove $\|X\|$. But we have the diagram:

$$\begin{array}{ccc} f_1 = 1 + \cdots + f_n = 1 & \dashrightarrow & X \\ \downarrow & & \downarrow \\ f_1 = 1 \vee \cdots \vee f_n = 1 & \longrightarrow & \|X\| \end{array}$$

where the dotted arrow merely exists because $f_1 = 1 \vee \cdots \vee f_n = 1$ has choice. But by lemma 3.2.2 we have that X is $(f_1 = 1 \vee \cdots \vee f_n = 1)$-local so we merely find a point in X. □

Proposition 3.2.5 Assume the presheaf topos satisfies dependent choice. Then the Zariski sheaf model satisfies dependent choice.

Proof A map between Zariski sheaf being surjective in the Zariski and presheaf model means the same thing by lemma 3.2.4. □

4 From Zariski sheaves to fppf sheaves

In this section we work in the Zariski topos.

4.1 Fppf sheaves in the Zariski topos

Definition 4.1.1 The fppf topology consists of $\text{Spec}(B)$ for all fppf boolean algebra B.

The checking that this is a topology is the same as in algebraic geometry.

Lemma 4.1.2 Any 2-module is flat.

Proof This is true for any discrete field, the idea is that any module is a filtered colimit of finitely presented modules, finitely presented modules over a discrete field are free therefore flat, and a filtered colimit of flat modules is flat. □

Remark 4.1.3 We think that any module over a boolean algebra is flat. We were not able to prove this yet.

Lemma 4.1.4 A boolean algebra B is fppf if and only if $0 \neq_B 1$.

Proof By lemma 4.1.2 we have that B is always flat, so we have to prove that:

$$\prod_{M:2-\text{Mod}} B \otimes M = 0 \to M = 0$$

if and only if:

$$0 \neq_B 1$$

The direct is clear, conversly if $0 \neq_B 1$ then:

$$2 \to B$$

is injective, but by lemma 4.1.2 we have that any M is flat so that:

$$M \to M \otimes B$$

is injective which is enough to conclude. □

Lemma 4.1.5 The fppf topology is subcanonical.

Proof We could just reuse the proof from SAG, but there probably are simpler ones. TODO □

Once again next theorem relies on the unproven theorem 2.0.2.

Theorem 4.1.6
The Fppf topos satisfy the following:

- For all c.p. B such that $0 \neq_B 1$, we have that $\|\text{Spec}(B)\|$
- For all c.p. boolean ring C, the map:

  $$C \to 2^{\text{Spec}(C)}$$

  is an equivalence.
- For all c.p. boolean ring C, the type $\text{Spec}(C)$ has representable choice.

Lemma 4.1.7 For all c.p. boolean algebras A, B, a map:

$$\text{Spec}(B) \to \text{Spec}(A)$$

is surjective if and only if the corresponding map:

$$A \to B$$

is injective.

Proof Assume a surjective map:

$$\text{Spec}(B) \to \text{Spec}(A)$$

corresponding to:

$$f : A \to B$$

Then given $a, b : A$ such that $f(a) = f(b)$ we have that:

$$\prod_{x:\text{Spec}(A)} x(a) = x(b)$$

indeed for any $x : \text{Spec}(A)$ we want to prove a proposition, so we can assume $y : \text{Spec}(B)$ such that $y \circ f = x$ and then $f(a) = f(b)$ allows us to conclude. Therefore $a = b$.

Conversely if the map is injective TODO (would be a consequence of all modules over boolean algebras being flat) □

4.2 Dependent choice from Zariski sheaves to Fppf sheaves

Lemma 4.2.1 Assume:

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

an fppf cover of affine scheme. If A is fppf, so is B.

Proof TODO easy □

Lemma 4.2.2 Assume given:

$$\cdots \to A_2 \to A_1 \to A_0$$

a tower of fppf covers, then:

$$\lim_i A_i \to A_0$$

is an fppf cover.

Proof It is enough to check that given a tower of maps of c.p. algebra:

$$B_0 \to B_1 \to B_2 \to \cdots$$

such that B_0 is fppf and the induced $\mathrm{Spec}(B_{i+1}) \to \mathrm{Spec}(B_i)$ is an fppf cover, we have that $\mathrm{colim}_i B_i$ is a c.p. fppf algebra. The c.p. part is clear, and we have $0 \neq_{B_i} 1$ for all i by inductively using lemma 4.2.1, so $0 = 1$ in $\mathrm{colim}_i B_i$ is a contradiction. □

Next lemma more or less that if dependent choice holds in the Zariski topos, it holds in the fppf topos:

Lemma 4.2.3 Assume given a tower of fppf sheaves:

$$\cdots \to X_2 \to X_1 \to X_0$$

where the maps are fppf surjective. Assuming dependent choice in the Zariski topos, we are able to show that the map:

$$\lim_i X_i \to X_0$$

is fppf surjective.

Proof Consider the tower:

$$\cdots \to M_2 \to M_1 \to M_0$$

where:

$$M_0 = X_0$$

and M_{n+1} is defined as the type of commuting dotted arrows such that:

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & X_0 \\ & \nwarrow & \uparrow \scriptstyle{fppf\ cover} \\ & & A_{n+1} \end{array}$$

Using fppf local choice, we can show that the maps $M_{n+1} \to M_n$ are surjective, so that by dependent choice $\lim_i M_i \to X_0$ is surjective. But given a point in its fiber means giving a commutative triangle:

$$\begin{array}{ccc} \lim_i X_i & \longrightarrow & X_0 \\ & \nwarrow & \uparrow \\ & & \lim_i A_i \end{array}$$

and we conclude by lemma 4.2.2. □