A Foundation for Synthetic Stone Duality

Felix Cherubini, Thierry Coquand, Freek Geerligs and Hugo Moeneclaey

December 3, 2024

Abstract

The language of homotopy type theory has proved to be appropriate as an internal language for various higher toposes, for example with Synthetic Algebraic Geometry for the Zariski topos. In this paper we apply such techniques to the higher topos corresponding to the light condensed sets of Dustin Clausen and Peter Scholze. This seems to be an appropriate setting to develop synthetic topology, similar to the work of Martín Escardó. To reason internally about light condensed sets, we use homotopy type theory extended with 4 axioms. Our axioms are strong enough to prove Markov's principle, LLPO and the negation of WLPO. We also define a type of open propositions, inducing a topology on any type. This leads to a synthetic topological study of (second countable) Stone and compact Hausdorff spaces. Indeed all functions are continuous in the sense that they respect this induced topology, and this topology is as expected for these class of types. For example, any map from the unit interval to itself is continuous in the usual epsilon-delta sense. We also use the synthetic homotopy theory given by the higher types of homotopy type theory to define and work with cohomology. As an application, we prove Brouwer's fixed-point theorem internally.

(This is a partially outdated version, which is kept because it contains statements and proofs which are not in the current, shorter version.)

Contents

Introduction

The language of homotopy type theory is a dependent type theory enriched with the univalence axiom and higher inductive types. It has proven exceptionnally well-suited to develop homotopy theory in a synthetic way [Pro13]. It also provides the precision needed to analyze categorical models of type theory [Wei24]. Moreover, the arguments in this language can be rather directly represented in proof assistants. We use homotopy type theory to give a synthetic development of topology, which is analogous to the work on synthetic algebraic geometry [CCH23].

We introduce four axioms which seem sufficient for expressing and proving basic notions of topology, based on the light condensed sets approach, introduced in [CS24]. Interestingly, this development establishes strong connections with constructive mathematics [BB85], particularly constructive reverse mathematics [Ish06; Die18]. Several of Brouwer's principles, such that any real function on the unit interval is continuous, or the celebrated fan theorem, are consequences of this system of axioms. Furthermore, we can also prove principles that are not intuitionistically valid, such as Markov's Principle, or even the so-called Lesser Limited Principle of Omniscience, a principle well studied in constructive reverse mathematics, which is *not* valid effectively.

This development also aligns closely with the program of Synthetic Topology [Esc04; Leš21]: there is a dominance of open propositions, providing any type with an intrinsic topology, and we capture in this way synthetically the notion of (second-countable) compact Hausdorff spaces. While working on this axiom system, we learnt about the related work [BC], which provides a different axiomatisation at the set level. We show that some of their axioms are consequences of our axiom system. In particular, we can introduce in our setting a notion of "Overtly Discrete" spaces, dual in some way to the notion of compact Hausforff spaces, like in Synthetic Topology[1].

A central theme of homotopy type theory is that the notion of *type* is more general than the notion of *set*. We illustrate this theme here as well: we can form in our setting the types of Stone spaces and of compact Hausdorff spaces (types which don't form a set but a groupoid), and show these types are closed under sigma types. It would be impossible to formulate such properties in the setting of simple type or set theory. Additionally, leveraging the elegant definition of cohomology groups in homotopy type theory [Pro13], which relies on higher types that are not sets, we prove, in a purely axiomatic way, a special case of a theorem of Dyckhoff [Dyc76], describing the cohomology of compact Hausdorff spaces. This characterisation also supports a type-theoretic proof of Brouwer's fixed point theorem, similar to the proof in [Shu18]. In our setting the theorem can be formulated in the usual way, and not in an approximated form.

It is important to stress that what we capture in this axiomatic way are the properties of light condensed sets that are *internally* valid. David Wärn [Wär24] has proved that an important property of abelian groups in the setting of light condensed sets, is *not* valid internally and thus cannot be proved in this axiomatic context. We believe however that our axiom system can be convenient for proving the results that are internally valid, as we hope is illustrated by the present paper. We also conjecture that the present axiom system is actually *complete* for the properties that are internally valid. Finally, we think that this system can be justified in a constructive metatheory using the work [CRS21].

Acknowledgements

The idea to use the topological characterization of stone spaces as totally disconnected, compact Hausdorff spaces to prove Theorem 4.3.7 was explained to us by Martín Escardó. More precisely, he gave us a proof of the following topological theorem, which we were able to adapt to our setting:

Theorem

[1] We actually have a derivation of their "directed univalence", but this will be presented in a following paper.

If $f : X \to S$ is a continuous map into a Stone space and all fibers of f are Stone spaces, then the topological space X is totally disconnected.

David Wärn noticed that Markov's principle (Theorem 1.4.2) holds.

1 Stone duality

1.1 Preliminaries

Definition 1.1.1 A type is countable iff it is merely equal to some decidable subset of $\mathbb{N}$.

Definition 1.1.2 A countably presented Boolean algebra B is a Boolean algebra such that there merely are countable sets I, J, a set of generators $g_i,\ i \in I$ and a set $f_j,\ j \in J$ of Boolean expressions over these generators such that B is equivalent to the quotient of the free Boolean algebra over the generators by the relations $f_j = 0$. We denote this algebra by $2[I]/(f_j)_{j:J}$.

Remark 1.1.3 Note that any countably presented algebra is also merely of the form $2[\mathbb{N}]/(r_n)_{n:\mathbb{N}}$.

Remark 1.1.4 We denote the type of countably presented Boolean algebras Boole. Note that this type does not depend on a choice of universe. Also note Boole has a natural category structure.

Example 1.1.5 If both the set of generators and relations are empty, we have the Boolean algebra 2. The underlying set is $\{0, 1\}$ and $0 \neq_2 1$. 2 is initial in Boole.

Definition 1.1.6 For B a countably presented Boolean algebra, we define $Sp(B)$ as the set of Boolean morphisms from B to 2. Any type which is merely equivalent to a type of the form $Sp(B)$ is called a Stone space.

Example 1.1.7 (i) There is only one Boolean map $2 \to 2$, thus $Sp(2)$ is the singleton type $\top$.

(ii) The tivial Boolean algebra is given by $2/(1)$. We have $0 = 1$ in the trivial Boolean algebra. As there cannot be a map from the trivial Boolean algebra into 2 preserving both $0, 1$, the corresponding Stone space is the empty type $\perp$.

(iii) We denote by C the Boolean algebra $2[\mathbb{N}]$. In this case $Sp(C)$ is Cantor space: $2^{\mathbb{N}}$, the set of binary sequences. If $\alpha : 2^{\mathbb{N}}$ and $n : \mathbb{N}$ we write $\alpha(n)$ for $\alpha(g_n)$.

(iv) We denote B_∞ for the Boolean algebra generated by $(g_n)_{n:\mathbb{N}}$ quotiented by the relations $g_m \wedge g_n = 0$ for $n \neq m$. A morphism $B_\infty \to 2$ corresponds to a function $\mathbb{N} \to 2$ that hits 1 at most once. We denote $Sp(B_\infty) = \mathbb{N}_\infty$. For $\alpha : \mathbb{N}_\infty$ and $n : \mathbb{N}$ we denote $\alpha(n)$ for $\alpha(g_n)$. By conjunctive normal form, any element of B_∞ can be written uniquely as $\bigvee_{i\in I} g_n$ or as $\bigwedge_{i\in I} \neg g_n$ for some finite $I \subseteq \mathbb{N}$. If $I = \emptyset$, then $\vee_{i\in I} g_i = 0, \bigwedge_{i\in I} \neg g_i = 1$.

Lemma 1.1.8 For $\alpha : 2^{\mathbb{N}}$, we have an equivalence of propositions:

$$(\forall_{n:\mathbb{N}} \alpha(n) = 0) \leftrightarrow Sp(2/(\alpha(n))_{n:\mathbb{N}}).$$

Proof There is at most one $x : 2 \to 2$, and it can only satisfy $x(\alpha(n)) = 0$ for all $n : \mathbb{N}$ iff $\alpha(n) \neq_2 1$ for all $n : \mathbb{N}$. As 2 has underlying set $\{0, 1\}$, we have $(\alpha(n) \neq_2 1) \to (\alpha(n) =_2 0)$. □

1.2 Axioms

Axiom 1 (Stone duality)
For any countably presented Boolean algebra B, the evaluation map $B \to 2^{Sp(B)}$ is an isomorphism.

Axiom 2 (Surjections are formal surjections)
For $g : B \to C$ a map in Boole, g is injective iff $(-) \circ g : Sp(C) \to Sp(B)$ is surjective.

Axiom 3 (Local choice)
Whenever we have $B : \mathsf{Boole}$, and some type family P over $Sp(B)$ with $\Pi_{s:Sp(B)} ||Ps||$, then there merely exists some $C : \mathsf{Boole}$ and surjection $q : Sp(C) \to Sp(B)$ with $\Pi_{t:Sp(C)} P(q(t))$.

Axiom 4 (Dependent choice)
Given a family of types $(E_n)_{n:\mathbb{N}}$ and a relation $R_n : E_n \to E_{n+1} \to \mathcal{U}$ such that for all n and $x : E_n$ there exists $y : E_{n+1}$ with $p : R_n\ x\ y$ then given $x_0 : E_0$ there exists $u : \Pi_{n:\mathbb{N}} E_n$ and $v : \Pi_{n:\mathbb{N}} R_n\ (u_n)\ (u_{n+1})$ and $u_0 = x_0$.

1.3 Anti-equivalence of Boole and Stone

By Axiom 1, Sp is an embedding from Boole to any universe of types. We denote it's image by Stone.

Remark 1.3.1 Stone types will take over the role of affine scheme from [CCH23], and we repeat some results here. Analogously to Lemma 3.1.2 of [CCH23], for X Stone, Stone duality tells us that $X = Sp(2^X)$. Proposition 2.2.1 of [CCH23] now says that Sp gives a natural equivalence

$$Hom_{\mathsf{Boole}}(A, B) = (Sp(B) \to Sp(A)) \tag{1}$$

Stone also has a natural category structure. By the above and Lemma 9.4.5 of [Pro13], the map Sp defines a dual equivalence of categories between Boole and Stone. In particular the spectrum of any colimit in Boole is the limit of the spectrum of the opposite diagram.

Remark 1.3.2 Local choice can also be formulated as follows: whenever we have $S : \mathsf{Stone}$, E, F arbitrary types, a map $f : S \to F$ and a surjection $e : E \twoheadrightarrow F$, there exists a Stone space T, a surjective map $T \twoheadrightarrow S$ and an arrow $T \to E$ making the following diagram commute:

$$\begin{tikzcd} T \arrow[r, dashed] \arrow[d, dashed, two heads] & E \arrow[d, two heads, "e"] \\ S \arrow[r, "f"] & F \end{tikzcd} \tag{2}$$

Lemma 1.3.3 For $B : \mathsf{Boole}$, we have $0 =_B 1$ iff $\neg Sp(B)$.

Proof If $0 =_B 1$, there is no map $B \to 2$ respecting both 0 and 1, thus $\neg Sp(B)$. Conversely, if $\neg Sp(B)$, then $Sp(B)$ equals $\bot$, the spectrum of the trivial Boolean algebra. As Sp is an embedding, B is equivalent to the trivial Boolean algebra, hence $0 =_B 1$. $\square$

Corollary 1.3.4 For $S : \mathsf{Stone}$, we have that $\neg\neg S \to ||S||$

Proof Let $B : \mathsf{Boole}$ and suppose $\neg\neg Sp(B)$. Let $f : 2 \to B$. If $f(0) = f(1)$ then $0 = 1$ in B, thus $\neg Sp(B)$, contradicting our assumption. Hence $f(0) \neq f(1)$. Hence by case distinction on 2 we can show that f is injective. By Axiom 2 the map $Sp(B) \to Sp(2)$ is surjective, thus $Sp(B)$ is merely inhabited. $\square$

1.4 Principles of omniscience

In constructive mathematics, we do not assume the law of excluded middle (LEM). There are some principles called principles of omniscience that are weaker than LEM, which can be used to describe how close a logical system is to satisfying LEM. References on these principles include [Die18; Ish06]. In this section, we will show that two of them (MP and LLPO) hold, and one (WLPO) fails in our system.

Theorem 1.4.1 (The negation of the weak lesser principle of omniscience ($\neg$WLPO))

$$\neg\forall_{\alpha:2^{\mathbb{N}}}((\forall_{n:\mathbb{N}}\alpha(n) = 0) \vee \neg(\forall_{n:\mathbb{N}}\alpha(n) = 0)) \tag{3}$$

Proof Let $f : 2^{\mathbb{N}} \to 2$ such that $f(\alpha) = 0$ iff $\forall_{n:\mathbb{N}}\alpha(n) = 0$. By Axiom 1, there is some $c : C$ with $f(\alpha) = 0 \leftrightarrow \alpha(c) = 0$. We can express c using finitely many generators $(g_n)_{n \leq N}$. Now consider $\beta, \gamma : 2^{\mathbb{N}}$ given by $\beta(g_n) = 0$ for all $n : \mathbb{N}$ and $\gamma(g_n) = 0$ iff $n \leq N$. As β, γ are equal on $(g_n)_{n \leq N}$, we have $\beta(c) = \gamma(c)$. However, $f(\beta) = 0$ and $f(\gamma) = 1$, giving a contradiction as required. $\square$

The following result is due to David Wärn:

Theorem 1.4.2 (Markov's principle (MP))
For $\alpha : \mathbb{N}_\infty$, we have that

$$(\neg(\forall_{n:\mathbb{N}}\alpha(n) = 0)) \to \Sigma_{n:\mathbb{N}}\alpha(n) = 1 \tag{4}$$

Proof By Lemma 1.1.8, we have that $\neg(\forall_{n:\mathbb{N}}\alpha(n) = 0)$ implies that $Sp(2/(\alpha(n))_{n:\mathbb{N}}$ is empty. Hence $2/(\alpha(n))_{n:\mathbb{N}}$ is trivial by Lemma 1.3.3. Then there is a finite subset $N_0 \subseteq \mathbb{N}$ with $\bigvee_{i:N_0} \alpha(i) = 1$. As $\alpha(i) \in \{0, 1\}$ and $\alpha(i) = 1$ for at most one $i : \mathbb{N}$, there exists an unique $n \in \mathbb{N}$ with $\alpha(n) = 1$. $\square$

Corollary 1.4.3 For $\alpha : 2^{\mathbb{N}}$, we have that

$$(\neg(\forall_{n:\mathbb{N}}\alpha(n) = 0)) \to \Sigma_{n:\mathbb{N}}\alpha(n) = 1 \tag{5}$$

Proof Given $\alpha : 2^{\mathbb{N}}$, consider the sequence $\alpha' : \mathbb{N}_\infty$ satisfying $\alpha'(n) = 1$ iff n is minimal with $\alpha(n) = 1$. Then apply the above theorem. $\square$

Theorem 1.4.4 (The lesser limited principle of omniscience (LLPO))
For $\alpha : \mathbb{N}_\infty$, we have that

$$\forall_{k:\mathbb{N}}\alpha(2k) = 0 \vee \forall_{k:\mathbb{N}}\alpha(2k+1) = 0 \tag{6}$$

Proof Define $f : B_\infty \to B_\infty \times B_\infty$ on generators as follows:

$$f(g_n) = \begin{cases} (g_k, 0) \text{ if } n = 2k \\ (0, g_k) \text{ if } n = 2k+1 \end{cases} \tag{7}$$

Note that f is well-defined as map in Boole as $f(g_n) \wedge f(g_m) = 0$ whenever $m \neq n$. We claim that f is injective. If $I \subseteq \mathbb{N}$, write $I_0 = \{k \in \mathbb{N} | 2k \in I\}, I_1 = \{k \in \mathbb{N} | 2k+1 \in I\}$. Recall that any $x : B_\infty$ is of the form $\bigvee_{i\in I} g_i$ or $\bigwedge_{i \in I} \neg g_i$.

- If $x = \bigvee_{i\in I} g_i$, then $f(x) = (\bigvee_{i\in I_0} g_i, \bigvee_{i\in I_1}(g_i))$. So if $f(x) = 0$, then $I_0 = I_1 = I = \emptyset$ and $x = 0$.
- Suppose $x = \bigwedge_{i\in I} \neg g_i$. Then $f(x) = (\bigwedge_{i\in I_0} \neg g_i, \bigwedge_{i\in I_1} \neg g_i)$, so $f(x) \neq 0$.

By Axiom 2, f corresponds to a surjection $s : \mathbb{N}_\infty + \mathbb{N}_\infty \to \mathbb{N}_\infty$. Thus for $\alpha : \mathbb{N}_\infty$, there exists some $x : \mathbb{N}_\infty + \mathbb{N}_\infty$ such that $sx = \alpha$. If $x = inl(\beta)$, for any $k : \mathbb{N}$, we have that

$$\alpha(2k+1) = s(x)(2k+1) = x(f(g_{2k+1})) = inl(\beta)(0, g_k) = \beta(0) = 0.$$

Similarly, if $x = inr(\beta)$, we have $\alpha(2k) = 0$ for all $k : \mathbb{N}$. Thus Equation (6) holds for α as required. $\square$

The surjection $s : \mathbb{N}_\infty + \mathbb{N}_\infty \to \mathbb{N}_\infty$ as above does not have a section as the following shows:

Lemma 1.4.5 The function f as in Equation (7) does not have a retraction.

Proof Suppose $r : B_\infty \times B_\infty \to B_\infty$ is a retraction of f. Then $r(0, g_k) = g_{2k+1}, r(g_k, 0) = g_{2k}$. Note that $r(0,1) : B_\infty$ is expressable using only finitely many generators $(g_n)_{n\leq N}$ Note that $r(0,1) \geq r(0, g_k) = g_{2k+1}$ for all $k : \mathbb{N}$. As a consequence, $r(0,1)$ cannot be of the form $\bigvee_{i\in I} g_i$ By similar reasoning so does $r(1,0)$.But this contradicts

$$r(0,1) \wedge r(1,0) = r((1,0) \wedge (0,1)) = r(0,0) = 0$$

Thus no retraction exists. $\square$

1.5 Open and closed propositions

In this section we will introduce a topology on the type of propositions, and study their logical properties. We think of open and closed propositions respectively as countable disjunctions and conjunctions of decidable propositions. Such a definition is universe-independent, and can be made internally.

Definition 1.5.1 We define the types Open, Closed of open and closed propositions as follows:

- A proposition P is open iff there merely exists some $\alpha : 2^{\mathbb{N}}$ such that $P \leftrightarrow \exists_{n:\mathbb{N}}\alpha(n) = 0$.
- A proposition P is closed iff there merely exists some $\alpha : 2^{\mathbb{N}}$ such that $P \leftrightarrow \forall_{n:\mathbb{N}}\alpha(n) = 0$.

Remark 1.5.2 The negation of an open proposition is closed, and by MP (Theorem 1.4.2), the negation of a closed proposition is open. Also by MP, we have $\neg\neg P \to P$ whenever P is open or closed. By the negation of WLPO (Theorem 1.4.1), not every closed proposition is decidable. Therefore, not every open proposition is decidable. Every decidable proposition is both open and closed.

Lemma 1.5.3 Closed propositions are closed under countable conjunctions.

Proof Let $(P_n)_{n:\mathbb{N}}$ be a countable family of closed propositions. By countable choice, for each $n : \mathbb{N}$ we have an $\alpha_n : 2^{\mathbb{N}}$ such that $P_n \leftrightarrow \forall_{m:\mathbb{N}}\alpha_n(m) = 0$. Consider a surjection $s : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Let

$$\beta(k) = \alpha_{\pi_0(s(k))}(\pi_1(s(k))).$$

Note that $\forall_{k:\mathbb{N}}\beta(k) = 0$ iff $\forall_{m,n:\mathbb{N}}\alpha_m(n) = 0$, which happens iff $\forall_{n:\mathbb{N}}P_n$. Hence the countable conjunction of closed propositions is closed. $\square$

Lemma 1.5.4 Open propositions are closed under countable disjunctions.

Proof Similar to the previous lemma. $\square$

Corollary 1.5.5 If a proposition is both open and closed, it is decidable.

Proof If P is open and closed, by Remark 1.5.2, $\neg P$ is open. By Lemma 1.5.4, it follows $P \vee \neg P$ is open, hence equivalent to $\neg\neg(P \vee \neg P)$ by Remark 1.5.2. As the latter proposition is provable, we may conclude P is decidable. $\square$

Lemma 1.5.6 Closed propositions are closed under finite disjunctions.

Proof This statement is equivalent to LLPO (Theorem 1.4.4)by Proposition 1.4.1 of [Die18]. $\square$

Lemma 1.5.7 For $(P_n)_{n:\mathbb{N}}$ a sequence of closed propositions, we have $\neg\forall_{n:\mathbb{N}} P_n \leftrightarrow \exists_{n:\mathbb{N}} \neg P_n$.

Proof It is always the case that $\exists_{n:\mathbb{N}} \neg P_n \to \neg\forall_{n:\mathbb{N}} P_n$. For the converse direction, note that $\neg\exists_{n:\mathbb{N}} \neg P_n(x) \to \forall_{n:\mathbb{N}} \neg\neg P_n(x)$. By Remark 1.5.2, $\neg\neg P_n(x) \leftrightarrow P_n(x)$ for all $n : \mathbb{N}$. It follows that $\neg\forall_{n:\mathbb{N}} P_n(x) \to \neg\neg\exists_{n:\mathbb{N}} \neg P_n(x)$. As $\exists_{n:\mathbb{N}} \neg P_n(x)$ is a countable disjunction of open propositions, it is open by Lemma 1.5.4 and thus equivalent to $\neg\neg\exists_{n:\mathbb{N}} \neg P_n(x)$ by Remark 1.5.2. We conclude that $\neg\forall_{n:\mathbb{N}} P_n \to \exists_{n:\mathbb{N}} \neg P_n$ as required. $\square$

Lemma 1.5.8 If P is open and Q is closed, $P \to Q$ is closed. Also, if P is closed and Q open, then $P \to Q$ is open.

Proof Assume P open and Q closed, the other proof is similar. Note that $(\neg P \vee Q) \to (P \to Q)$ and $(P \to Q) \to \neg\neg(\neg P \vee Q)$. By Remark 1.5.2 it follows that $(\neg P \vee Q) \leftrightarrow (P \to Q)$, and using Lemma 1.5.6, we can conclude that $P \to Q$ is closed. $\square$

1.6 Types as spaces

Definition 1.6.1 Let X be a type. A subtype of X is a function $U : X \to \mathrm{Prop}$ to the type of propositions. We write $U \subseteq X$ to indicate that U is as above. If X is a set, a subtype may be called subset for emphasis. For subtypes $A, B \subseteq X$, we write $A \subseteq B$ as a shorthand for pointwise implication. We will freely switch between subtypes $U : X \to \mathrm{Prop}$ and the corresponding embeddings

$$\textstyle\sum_{x:X} U(x) \longrightarrow X \ .$$

In particular, if we write $x : U$ for a subtype $U : X \to \mathrm{Prop}$, we mean that $x : \sum_{x:X} U(x)$ – but we might silently project x to X. We will also denote $x \in U$ or $U(x)$ if we know that $x : X$.

The subobject Open of the type of propositions induces a topology on every type. This is the viewpoint taken in synthetic topology. We will follow the terminology of [Esc04; Leš21].

Definition 1.6.2 Let T be a type, and let $A \subseteq T$ be a subtype. We call $A \subseteq T$ open or closed iff $A(t)$ is open or closed respectively for all $t : T$.

Remark 1.6.3 It follows immediately that the pre-image of an open by any map of types sends is open, so that any map is continuous. In Theorem 3.3.1, we shall see that the resulting topology is as expected for second countable Stone spaces. In Remark 5.3.6, we shall see that the same for the unit interval.

2 Overtly discrete spaces

(Change the ι notation to have the domain on top, and π have codomain at bottom So $\pi^m_n \circ \pi_m = \pi_n, \iota^n_m \circ \iota_n = \iota_m$ $\iota^n_m : B_n \to B_m$.) (Discussion on what the colimit is exactly, refer to definition in[SDR20])

Definition 2.0.1 We call a type overtly discrete iff it is a sequential colimit of finite sets.

Remark 2.0.2 It follows from Corollary 7.7 of [SDR20] that overtly discrete types are sets. Note that the type of overtly discrete types is independent on choice of universe, so we can write ODisc for this type. If $B : \mathsf{ODisc}$, we will denote B_n for the objects of the underlying sequence and $\iota^n)m : B_n \to B_m, \iota_n : B_n \to B$ for the obvious maps.

2.1 Maps of overtly discrete types

Lemma 2.1.1 (Compactness of finite sets) Exponentiation by a finite sets commute with sequential colimit.

Proof (Reference to standard proof working here as well.) □

Remark 2.1.2 In the above proof, we used that any element $b \in B$ already occurs in some B_n. However, it does not necessarily occur uniquely in B_n. In general, B is overtly discrete and there exist some B_n with two elements corresponding to the same element in B, Theorem 7.4 from [SDR20] says that there merely exists some $m \geq n$ such that these elements become equal in B_m.

Lemma 2.1.3 (Any map between overtly discrete sets is a sequential colimit of maps between finite sets.)

Let B, C be overtly discrete, and let $f : B \to C$. There exists $(\mathbb{N}, \leq)$-indexed sequences of finite sets $(B_n)_{n:\mathbb{N}}, (C_n)_{n:\mathbb{N}}$ with colimits B, C respectively and compatible maps $f_n : B_n \to C_n$, such that f is the induced morphism $B \to C$.

Proof Let $(B_n)_{n:\mathbb{N}}, (C_n)_{n:\mathbb{N}}$ be sequences of finite sets with colimits B and C. Using Axiom 4, we will construct an increasing sequence of natural numbers n_i with a family of maps $f_i : B_i \to C_{n_i}$ such that the following diagram commutes for all $i > 0$. :

$$\begin{tikzcd}
B_{i-1} \arrow[r, "\iota^i_{i-1}"] \arrow[d, "f_{i-1}"] & B_i \arrow[r, "\iota_i"] \arrow[d, "f_i"] & B \arrow[d, "f"] \\
C_{n_{i-1}} \arrow[r, "\kappa^i_{i-1}"] & C_{n_i} \arrow[r, "\kappa_i"] & C
\end{tikzcd} \tag{8}$$

Suppose we have an initial segment $(n_i)_{i<k}$ of such a sequence with maps $(f_i)_{i<k}$ making Equation (8) commute for $i < k$. We shall show that in this case there exist $n_k : \mathbb{N}, f_k : B_k \to C_{n_k}$ extending it. Consider the map $f \circ \iota_k : B_k \to C$. As B_k is finite, Lemma 2.1.1 gives some $n'_k : \mathbb{N}$ such that it factors over some $C_{n'_k}$. Both f'_k, f_{k-1} induce the same map $B_{k-1} \to C$. As B_{k-1} is finite, from Remark 2.1.2 there is some $n_k \geq n'_k$ such that they become equal in C_{n_k}, and we have $f_k : B_k \to C_{n_k}$ such that the following does commute; by Axiom 4 we then get compatible maps as required.

$$\begin{tikzcd}
B_{k-1} \arrow[r] \arrow[d, "f_{k-1}"] & B_k \arrow[rr, "\iota_k"] \arrow[dr, "f_k"] & & B \arrow[d, "f"] \\
C_{n_{k-1}} \arrow[r] & C_{n'_k} \arrow[r] & C_{n_k} \arrow[r] & C
\end{tikzcd} \tag{9}$$

□

Lemma 2.1.4 (Denote $\Im(f)$ instead of $f(A)$.) (Can this be a shorter remark?) Let $f : A_\infty \to B_\infty$ be a map between overtly discrete types, and suppose we have $f_n : A_n \to B_n$ such that the following diagram commutes:

$$\begin{tikzcd}
A_n \arrow[r, "\iota^m_n"] \arrow[d, "f_n"] & A_m \arrow[r, "\iota^\infty_m"] \arrow[d, "f_m"] & A_\infty \arrow[d, "f"] \\
B_n \arrow[r, "\kappa^m_n"'] & B_m \arrow[r, "\kappa^\infty_m"'] & B_\infty
\end{tikzcd} \tag{10}$$

Then $f(A)$ is the colimit of $f_n(A_n)$, and the maps $A \twoheadrightarrow f(A)$ and $f(A) \hookrightarrow B$ are induced by the maps $A_n \twoheadrightarrow f_n(A_n)$ and $f_n(A_n) \hookrightarrow B_n$ respectively.

Proof For $n \leq m$, we have that $\kappa^m_n(f_n(A_n)) = f_m(\iota^m_n(A_n)) \subseteq f_m(A_m)$, hence we can take the corestriction of the map $f_n(A_n) \to B_m$ to $f_m(A_m)$ to get maps $\lambda^m_n : f_n(A_n) \to f_m(A_m)$ making the following diagram commute:

$$\begin{tikzcd}
A_n \arrow[r, "\iota^m_n"] \arrow[d, two heads, "e_n"] & A_m \arrow[r, "\iota^\infty_m"] \arrow[d, two heads, "e_m"] & A_\infty \arrow[d, two heads, "e_\infty"] \\
f_n(A_n) \arrow[r, dashed, "\lambda^m_n"] \arrow[d, hook, "i_n"] & f_m(A_m) \arrow[r, dashed, "\lambda^\infty_m"] \arrow[d, hook, "i_m"] & f(A_\infty) \arrow[d, hook, "i_\infty"] \\
B_n \arrow[r, "\kappa^m_n"'] & B_m \arrow[r, "\kappa^\infty_m"'] & B_\infty
\end{tikzcd} \tag{11}$$

Also it is clear that any $b : f(A_\infty)$ already occurs in some $f_n(A_n)$, hence $f(A_\infty)$ is colimiting. □

Corollary 2.1.5 In Lemma 2.1.3, when f is injective or surjective, we can choose presentations such that each f_n is also injective or surjective respectively.

Proof Using Lemma 2.1.3 and Lemma 2.1.4, we get a factorization as in Equation (11). If f is injective, then e is an isomorphism. Hence A is the colimit of $f_n(A_n)$, and we can take $f'_n = i_n$. Similarly, if f is surjective i is an isomorphism and we consider B as colimit of $f_n(A_n)$ and take $f'_n = e_n$. $\square$

2.2 Closure properties of ODisc

Remark 2.2.1 As sequential colimits commute with finite colimits, and finite sets are closed under finite colimits, ODisc is closed under finite colimits as well.

Lemma 2.2.2 The colimit an $(\mathbb{N}, \leq)$-indexed sequence overtly discrete types is overtly discrete.

Proof By applying Axiom 4 to Lemma 2.1.3, given a colimit of the sequence A_i, we can find a quarter-plane of the form

$$\begin{tikzcd} A_{0,0} \arrow[r] \arrow[d] & A_{0,1} \arrow[r] \arrow[d] & \cdots \\ A_{1,0} \arrow[r] \arrow[d] & A_{1,1} \arrow[r] \arrow[d] & \cdots \\ \vdots & \vdots & \ddots \end{tikzcd} \tag{12}$$

where all the $A_{i,j}$ are finite sets, and A_i is the colimit in j of $A_{i,j}$ and the maps $A_i \to A_k$ are induced by maps $A_{i,j} \to A_{k,j}$. The colimit of the above quarter-plane is also the colimit of the induced $(\mathbb{N}, \leq)$-indexed sequence $A_{j,j}$, which is overtly discrete by definition. $\square$

Corollary 2.2.3 Overtly discrete types are closed under Σ.

Proof Let B be overtly discrete and $X : B \to \mathcal{U}$ be a B-indexed family of overtly discrete types. For any $i : \mathbb{N}$, we have a finite coproduct of overtly discrete types $\Sigma_{b:B_i}(X \circ \iota_i(b))$. As colimits commute with finite coproducts, this is overtly discrete. By Theorem 5.1 of [SDR20], taking the colimit in i, we get $\Sigma_{b:B} X(b)$. By the above Lemma, this is overtly discrete. $\square$

Remark 2.2.4 Note that the sequential colimit commutes with the propositional truncation, thus for B : ODisc, we have $||B||$: ODisc.

2.3 Open and ODisc

Lemma 2.3.1 Whenever P is a proposition and overtly discrete, P is open.

Proof If P is overtly discrete, then $P \leftrightarrow \exists_{n:\mathbb{N}} P_n$. As every P_n is finite, it is decidable. Hence P is a countable disjunction of decidable propositions, hence open.

Lemma 2.3.2 Whenever P is a an open proposition, it is overtly discrete.

Proof Suppose $P \leftrightarrow \exists_{n:\mathbb{N}} \alpha_n = 1$. Let $P_n = \exists_{k \leq n}(\alpha_k = 1)$, which is a decidable proposition, hence a finite set. Then the colimit of P_n is P. $\square$

Corollary 2.3.3 A proposition is open iff it is overtly discrete.

Proof Immediate by the above two lemmas. $\square$

Corollary 2.3.4 Open propositions are closed under dependent sums.

Proof Immediate from Corollary 2.2.3 and Corollary 2.3.3. $\square$

Corollary 2.3.5 (transitivity of openness) Let T be a type, let $V \subseteq T$ open and let $W \subseteq V$ open. Then the composite $W \subseteq V \subseteq T$ is open as well.

Proof Denote $W' \subseteq T$ for the composite. Note that $W'(t) = \Sigma_{v:V(t)} W(t,v)$. As open propositions are closed under dependent sums (Corollary 2.3.4), $W'(t)$ is an open proposition, as required. $\square$

Remark 2.3.6 As the true proposition is open and openness is transitive, Open can be called a dominance according to Proposition 2.25 of [Leš21]

Lemma 2.3.7 A type B is overtly discrete iff it is the quotient of a countable set by an open equivalence relation.

Proof Is $B : \mathsf{ODisc}$, then $B = (\Sigma_{n:\mathbb{N}} B_n)/\sim_B$ where $\sim_B$ is the reflexive closure of $(n,b) \sim (m, \iota_n^m b)$ for $n \leq m$. Conversely, assume $B = D/R$ with $D \subseteq \mathbb{N}$ decidable and R open. By countable choice we get $\alpha_{(\cdot,\cdot)} : D \to D \to 2^{\mathbb{N}}$ such that $R(x,y) \leftrightarrow \exists_{n:\mathbb{N}} \alpha(n) = 1$. Define $D_n = (D \cap \mathbb{N}_{\leq n})$, and $R_n : D_n \to D_n \to 2$ so that $R_n(x,y)$ is the equivalence relation induced by the relation that checks whether $\alpha_{(x,y)}(k) = 1$ for some $k \leq n$. Note that $B_n = D_n/R_n$ is a finite set, and has colimit B. $\square$

Lemma 2.3.8 For any open $U \subseteq \mathbb{N}$, there merely exists a decidable set D in $\mathbb{N}$ such that $\Sigma_{n:\mathbb{N}} D(n) \simeq \Sigma_{n:\mathbb{N}} U(n)$.

Proof Using countable choice, we get a map $\alpha_{(\cdot)} : \mathbb{N} \to \mathbb{N}_\infty$ such that $U(n) \leftrightarrow \Sigma_{k:\mathbb{N}} \alpha_n(k) = 1$. Hence $\Sigma_{n:\mathbb{N}} U(n) \simeq \Sigma_{n,k:\mathbb{N}} (\alpha_n(k) = 0)$ using $\mathbb{N} = \mathbb{N} \times \mathbb{N}$, we can conclude. $\square$

2.4 Relating ODisc and Boole

Lemma 2.4.1 Every countably presented Boolean algebra can be seen as the colimit of a sequence of finite Boolean algebras.

Proof Consider a countably presented Boolean algebra of the form $B = 2[\mathbb{N}]/(r_n)_{n:\mathbb{N}}$. For each $n : \mathbb{N}$, let G_n be the union of $\{g_i | i \leq n\}$ and the finite set of terms occurring in $(r_i)_{i \leq n}$. Denote $B_n = 2[G_n]/(r_i)_{i \leq n}$. Each B_n is a finite Boolean algebra, and there are canonical maps $B_n \to B_{n+1}$. We claim that B is the colimit of this sequence. $\square$

Corollary 2.4.2 A Boolean algebra B is overtly discrete iff it is countably presented.

Proof Assume $B : \mathsf{ODisc}$. By Lemma 2.3.7, B has open equality. Also $F = 2[\Sigma_{n:\mathbb{N}} B_n]$ is countable and we have a canonical Boolean morphism $f : F \to B$. By countable choice, we get for each $a, b : F$ a sequence $\alpha_{(a,b)} : 2^{\mathbb{N}}$ such that $(f(a) = f(b)) \leftrightarrow \exists_{k:\mathbb{N}} (\alpha_{(a,b)} k = 1)$. Consider $r : F \times F \times \mathbb{N} \to F$ given by

$$r(a,b,k) = \begin{cases} a - b & \text{if } \alpha_{(a,b)}(k) = 1 \\ 0 & \text{if } \alpha_{(a,b)}(k) = 0 \end{cases}$$

Then $B = F/(r(a,b,n))_{(a,b,n):F \times F \times \mathbb{N}}$. The converse direction was shown in Lemma 2.4.1. $\square$

Remark 2.4.3 By Lemma 2.3.7, Corollary 2.2.3 and Corollary 2.4.2, it follows that any $g : B \to C$ in Boole has an overtly discrete kernel. As a consequence, the kernel is countable and $B/Ker(g)$ is in Boole. By uniqueness of epi-mono factorizations and Axiom 2, the factorization $B \twoheadrightarrow B/Ker(g) \hookrightarrow C$ corresponds to $Sp(C) \twoheadrightarrow Sp(B/Ker(g)) \hookrightarrow Sp(B)$.

3 Stone spaces

3.1 Stone spaces as profinite sets

Here we present Stone spaces as limits of $(\mathbb{N}, \geq)$-indexed sequences of finite sets. This is the perspective taken in Condensed Mathematics [Sch19; Ásg21; CS24]. Some of the results in this section are specific versions of the axioms used in [BC]. A full generalization is part of future work.

Lemma 3.1.1 Any $S : \mathsf{Stone}$ can be described as the limit of some $(\mathbb{N}, \geq)$-indexed sequence of finite sets.

Proof By Remark 1.3.1, Lemma 2.4.1 and Lemma 2.2.2, for $B : \mathsf{Boole}$, we have $Sp(B)$ the limit of $Sp(B_n)$, which are finite sets. $\square$

Lemma 3.1.2 The limit of some $(\mathbb{N}, \geq)$-indexed sequence S_n of finite sets is a Stone space.

Proof For finite sets, we have that $Sp(2^{S_n}) = S_n$, hence each S_n is Stone. By Remark 1.3.1, Lemma 2.4.1 and Lemma 2.2.2, Stone is closed under sequential limits. $\square$

Remark 3.1.3 Whenever $S : \mathsf{Stone}$, we shall denote S_n for the underlying sequence and whenever $n \leq m$, we denote π^n_m for the maps $S_m \to S_n$, and $\pi_n : S \to S_n$.

Remark 3.1.4 Dually to Remark 2.2.1 and Lemma 2.2.2, Stone spaces are closed under finite limits and sequential limits. By Corollary 2.1.5 and Axiom 2 when we have a map of Stone spaces $f : S \to T$, we have $(\mathbb{N}, \geq)$-indexed sequences S_n, T_n with limits S and T respectively and maps $f_n : S_n \to T_n$ inducing f, and if f is surjective or injective, we can choose all f_n to be surjective or injective respectively as well.

Lemma 3.1.5 For $S : \mathsf{Stone}, k : \mathbb{N}$ we have that $Fin(k)^S$ is the colimit of $Fin(k)^{S_n}$.

Proof By Remark 1.3.1 we have $Fin(k)^S = (2^S)^{2^{Fin(k)}}$. Note that $2^F in(k)$ is finite, thus by Lemma 2.1.1, the latter is the colimit of $(2^{S_n})^{2^F in(k)}$. By applying Remark 1.3.1 again, these types are $Fin(k)^S_n$ as required. $\square$

Lemma 3.1.6 For $S : \mathsf{Stone}$ and $f : S \to \mathbb{N}$, there merely exists some $N : \mathbb{N}$ with $f(S) \subseteq \mathbb{N}_{\leq N}$.

Proof For each $n : \mathbb{N}$, the fiber of f over n is a decidable subset $f_n : S \to 2$. We must have that $Sp(2^S/(f_n)_{n:\mathbb{N}}) = \bot$, hence there exists some $N : \mathbb{N}$ with $\bigvee_{n \leq N} f_n =_{2^S} 1$. It follows that $f(s) \leq N$ for all $s : S$ as required. $\square$

Corollary 3.1.7 For $S : \mathsf{Stone}$, we have that $\mathbb{N}^S$ is the colimit of $\mathbb{N}^{S_n}$.

Proof By Lemma 3.1.6 we have that any map $S \to \mathbb{N}$ factors as $S \to Fin(k) \hookrightarrow \mathbb{N}$ for some $k : \mathbb{N}$. By Lemma 3.1.5, such a map is uniquely determined by compatible maps $S_n \to Fin(k)$, hence by compatible maps $S_n \to \mathbb{N}$, as required. $\square$

3.2 Closed and Stone

Corollary 3.2.1 Whenever $S : \mathsf{Stone}$, $||S||$ is closed.

Proof By Lemma 1.3.3, $\neg S$ is equivalent to $0 =_B 1$, which is open by the above. Hence $\neg\neg S$ is a closed proposition, and by Corollary 1.3.4, so is $||S||$. $\square$

Corollary 3.2.2 A proposition P is closed iff it is a Stone space.

Proof By the above, if S is both a Stone space and a proposition, it is closed. By Lemma 1.1.8, any closed proposition is Stone. $\square$

Lemma 3.2.3 Whenever $S : \mathsf{Stone}$, and $s, t : S$, the proposition $s = t$ is closed.

Proof Suppose $S = Sp(B)$ and let G be the generators of B. Note that $s = t$ iff $s(g) =_2 t(g)$ for all $g : G$. As G is countable, and equality in 2 is decidable, $s = t$ is a countable conjunction of decidable propositions, hence closed. $\square$

The following question was asked by Bas Spitters at TYPES 2024:

Corollary 3.2.4 For $S : \mathsf{Stone}$ and $x, y, z : S$

$$x \neq y \to (x \neq z \vee y \neq z) \tag{13}$$

Proof As $x \neq y$, we can show that $\neg(x = z \wedge y = z)$. This in turn implies $\neg\neg(x \neq z \vee y \neq z)$. As, $x \neq z$ and $y \neq z$ are both open propositions, by Lemma 1.5.4 so is their disjunction. By Remark 1.5.2, that disjunction is double negation stable and Equation (13) follows. $\square$

Remark 3.2.5 If Equation (13) holds in a type, we say that it's inequality is an apartness relation. By a similar proof as above, it can be shown that in our setting inequality is an apartness relation as soon as equality is open or closed.

3.3 The topology on Stone spaces

Theorem 3.3.1
Let $A \subseteq S$ be a subset of a Stone space. TFAE:

(i) There exists a map $\alpha_{(\cdot)} : S \to 2^{\mathbb{N}}$ such that $A(x) \leftrightarrow \forall_{n:\mathbb{N}} \alpha_x(n) = 0$ for any $x : S$.

(ii) There exists some countable family $D_n,\ n : \mathbb{N}$ of decidable subsets of S with $A = \bigcap_{n:\mathbb{N}} D_n$.

(iii) There exists a Stone space T and some embedding $T \to S$ which image is A

(iv) There exists a Stone space T and some map $T \to S$ which image is A.

(v) A is closed.

Proof

$(i) \leftrightarrow (ii)$. D_n and $\alpha_{(\cdot)}$ can be defined from each other by $D_n(x) \leftrightarrow (\alpha_x(n) = 0)$. Then observe that

$$(\bigcap_{n:\mathbb{N}} D_n)(x) \leftrightarrow \forall_{n:\mathbb{N}}(\alpha_x(n) = 0) \tag{14}$$

$(ii) \to (iii)$. Let $S = Sp(B)$. By Axiom 1, we have $d_n,\ n : \mathbb{N}$ terms of B such that $D_n = \{x : S | x(d_n) = 0\}$. Let $C = B/(d_n)_{n:\mathbb{N}}$. Then the map $Sp(C) \to S$ is as desired because

$$Sp(C) = \{x : S | \forall_{n:\mathbb{N}} x(d_n) = 0\} = \bigcap_{n:\mathbb{N}} D_n.$$

$(iii) \to (iv)$ Immediate.

$(iv) \to (i)$. Assume $f : T \to S$ corresponds to $g : B \to C$ in Boole. By Remark 2.4.3, $f(T) = Sp(B/Ker(g))$, and there is a surjection $d. : \mathbb{N} \to Ker(g)$. For D_n corresponding to d_n, we have $Sp(B/Ker(g)) = \bigcap_{n:\mathbb{N}} D_n$.

$(i) \to (v)$. By definition.

$(v) \to (iv)$. As A is closed, it corresponds to a map $a : S \to \mathsf{Closed}$. We can cover the closed propositions with Cantor space by sending $\alpha \mapsto \forall_{n:\mathbb{N}} \alpha(n) = 0$. By Axiom 3 gives us that there merely exists $T, e, \beta.$ as follows:

$$\begin{array}{ccc} T & \xrightarrow{\beta.} & 2^{\mathbb{N}} \\ \big\downarrow e & & \big\downarrow \forall_{n:\mathbb{N}}(\cdot)n=0 \\ S & \xrightarrow{a} & \mathsf{Closed} \end{array} \tag{15}$$

Define $B(x) \leftrightarrow \forall_{n:\mathbb{N}} \beta_x(n) = 0$. As $(i) \to (iii)$ by the above, B is the image of some Stone space. Furthermore, note that A is the image of B, thus A is the image of some Stone space. ☐

Remark 3.3.2 Using condition (iii), the previous result implies that closed subtype of Stone spaces are Stone.

Corollary 3.3.3 For $S : \mathsf{Stone}$ and $A \subseteq S$ closed, we have $\exists_{x:S} A(x)$ is closed.

Proof By Remark 3.3.2 we have that $\Sigma_{x:S} A(x)$ is Stone, so its truncation is closed by Corollary 3.2.1.☐

Corollary 3.3.4 Closed propositions are closed under dependent sums.

Proof Let $P : \mathsf{Closed}$ and $Q : P \to \mathsf{Closed}$. Then $\Sigma_{p:P} Q(p) \leftrightarrow \exists_{p:P} Q(p)$. As P is Stone by Corollary 3.2.2, Corollary 3.3.3 gives that $\Sigma_{p:P} Q(p)$ is closed. ☐

Remark 3.3.5 Analogously to Corollary 2.3.5 and Remark 2.3.6, it follows that closedness is transitive and Closed forms a dominance.

Lemma 3.3.6 If $S : \mathsf{Stone}$, and $F, G : S \to \mathsf{Closed}$ be such that $F \cap G = \emptyset$. Then there exists a decidable subset $D : S \to 2$ such $F \subseteq D, G \subseteq \neg D$.

Proof (Too shorten this (and some other proofs), I've removed some negations and pretended $D : S \to 2$ is given by $\{x : S | x(d) = 0\}$ instead of $\{x : S | x(d) = 1\}$) Assume $S = Sp(B)$. By Theorem 3.3.1, there exists sequences $f_n, g_n : B,\ n : \mathbb{N}$ such that $x \in F$ iff $x(f_n) = 0$ for all $n : \mathbb{N}$ and $y \in G$ iff $y(g_m) = 0$ for all $m : \mathbb{N}$. Denote $R \subseteq B$ for $\{f_n | n : \mathbb{N}\} \cup \{g_n | n : \mathbb{N}\}$. Note that $Sp(B/R) = F \cap G = \emptyset$, so by Lemma 1.3.3 there exists finite sets $I, J \subseteq \mathbb{N}$ such that $1 =_B ((\bigvee_{i \in I} f_i) \vee (\bigvee_{j \in J} g_j))$. Let $y \in F$, then $y(f_i) = 0$ for all $i \in I$, hence $y(\bigvee_{j \in J} g_j) = 1$ And if $x \in G$, we have $x(\bigvee_{j \in J} g_j) = 0$. Thus we can define the required D by $D(x) \leftrightarrow x(\bigvee_{j \in J} g_j) = 0$.

4 Compact Hausdorff spaces

Definition 4.0.1 A type X is called compact Hausdorff if there exists some $S : \mathsf{Stone}$ and some equivalence relation $\sim\ : S \times S \to \mathsf{Closed}$ such that $X \simeq S/\sim$. We denote CHaus for the type of compact Hausdorff types.

4.1 Topology on compact Hausdorff spaces

Lemma 4.1.1 Let $X : \mathsf{CHaus}$ be given as $X = S/\sim$ with quotient map $q : S \twoheadrightarrow X$. Then $A \subseteq X$ is closed if and only if it is the image of a closed in S under q.

Proof As q is surjective, we have $q(q^{-1}(A)) = A$. If A is closed, so is $q^{-1}(A)$ and hence A is the image of a closed subtype of S. Conversely, let $B \subseteq S$ be closed. Define $A' \subseteq S$ by

$$A'(s) := \exists_{s:S}(B(t) \wedge s \sim t).$$

As $B, \sim$ are closed, by Lemma 1.5.3 and Corollary 3.3.3, we have that A' is closed. Also A' respects $\sim$, hence induces a map $A : X \to \mathsf{Closed}$. Furthermore, $A'(q(s))$ iff $q(s) \in q(B)$. Hence $A = q(B)$. □

Remark 4.1.2 Let $X : \mathsf{CHaus}$. From Theorem 3.3.1, it follows that $A \subseteq X$ is closed iff it is the image of a map $T \to X$ for some $T : \mathsf{Stone}$. If A is closed, from Corollary 3.3.3, it follows that $\exists_{x:X} A(x)$ is closed as well.

Corollary 4.1.3 For $U \subseteq X$ an open subset of a compact Hausdorff space, we have $\forall_{x:X} U(x)$ open.

Proof As U is open, $\neg U$ is closed. By Remark 4.1.2 $\exists_{x:X} \neg U(x)$ is closed. Using Remark 1.5.2, it follows that $\neg(\exists_{x:X} \neg U(x))$ is open. Furthermore, it is equivalent to $\forall_{x:X} \neg\neg U(x)$, which is equivalent to $\forall_{x:X} U(x)$ by Remark 1.5.2. □

Lemma 4.1.4 Whenever $X : \mathsf{CHaus}$ and $C_n : X \to \mathsf{Closed}$ closed subsets with $\bigcap_{n:\mathbb{N}} C_n = \emptyset$, there is some $N : \mathbb{N}$ with $\bigcap_{n \leq N} C_n = \emptyset$.

Proof By Theorem 3.3.1, and Lemma 4.1.1 it is sufficient to prove the above when X is Stone and C_n decidable. So assume $X = Sp(B)$ and $c_n : B$ are such that $C_n = \{x : B \to 2 | x(c_n) = 1\}$. Then the set of maps $B \to 2$ sending all c_n to 1 is given by

$$Sp(B/\{\neg c_n | n : \mathbb{N}\}) \simeq \bigcap_{n:\mathbb{N}} C_n = \emptyset.$$

Hence $0 = 1$ in $B/(\neg c_n)_{n:\mathbb{N}}$, and there is some $N : \mathbb{N}$ with $\bigvee_{n \leq N}(\neg c_n) = 1$, which also means that

$$\emptyset = Sp(B/(\neg c_n)_{n \leq N}) \simeq \bigcap_{n \leq N} C_n.$$

Corollary 4.1.5 Let $X, Y : \mathsf{CHaus}$ and $f : Y \to X$. Suppose $(G_n)_{n:\mathbb{N}}$ is a decreasing sequence of closed subsets of Y. Then $f(\bigcap_{n:\mathbb{N}} G_n) = \bigcap_{n:\mathbb{N}}(f(G_n))$.

Proof It is always the case that $f(\bigcap_{n:\mathbb{N}} G_n) \subseteq \bigcap_{n:\mathbb{N}}(f(G_n))$. For the converse direction, suppose that $x \in f(G_n)$ for all $n : \mathbb{N}$. Define $F : Y \to \mathsf{Closed}$ by $F(y) := (f(y) = x)$. F defines a closed subset, furthermore, $F \cap G_n \neq \emptyset$ for all $n : \mathbb{N}$. Thus $\bigcap_{n:\mathbb{N}}(F \cap G_n) \neq \emptyset$ by Lemma 4.1.4. By Remark 4.1.2 and Remark 1.5.2, there merely exists some y in $\bigcap_{n:\mathbb{N}}(F \cap G_n)$. Thus $x \in f(\bigcap_{n:\mathbb{N}} G_n)$ as required. □

Corollary 4.1.6 Let $A \subseteq X$ be a subtype of a compact Hausdorff space. Let $p : S \to X$ be any presentation of X with $S : \mathsf{Stone}$. Then:

- A is closed iff it can be written as $\bigcap_{n:\mathbb{N}} p(D_n)$ for some sequence $D_n \subseteq S$ decidable.
- A is open iff it can be written as $\bigcup_{n:\mathbb{N}} \neg p(D_n)$ for some sequence $D_n \subseteq S$ decidable.

Proof The characterization of closed sets follows from characterization (ii) in Theorem 3.3.1, Lemma 4.1.1 and Corollary 4.1.5. The characterization of open sets then follows from Remark 1.5.2 and Lemma 1.5.7. □

Corollary 4.1.7 Any $X : \mathsf{CHaus}$ is second countable (has a topological basis which is countable).

Proof By Corollary 4.1.6, a basis is given by decidable subsets of some $S : \mathsf{Stone}$. By Stone duality, such a basis forms a countably presented Boolean algebra, which is countable. □

Lemma 4.1.8 Let $X : \mathsf{CHaus}$, and let $A, B : X \to \mathsf{Closed}$ be disjoint. Then there exist $U, V : X \to \mathsf{Open}$ disjoint with $A \subseteq U, B \subseteq V$, and $B \cap U = A \cap V = \emptyset$.

Proof Let $q : S \to X$ be a projection map presenting X. As $q^{-1}(A), q^{-1}(B)$ are closed, by Lemma 3.3.6, there is some $D : S \to 2$ such that $q^{-1}(A) \subseteq D, q^{-1}(B) \subseteq \neg D$. Note that $q(D), q(\neg D)$ are closed by Lemma 4.1.1. Furthermore, as $q^{-1}(A) \cap \neg D = \emptyset$, we have that $A \subseteq \neg q(\neg D)$. As $A \cap B = \emptyset$, we have that $A \subseteq \neg q(\neg D) \cap \neg B := U$. Similarly, $B \subseteq \neg q(D) \cap \neg A := V$. By definition, U, V are as required. □

4.2 Compact Hausdorff spaces are stable under dependent sums

Lemma 4.2.1 A type X is Stone iff it is merely a closed in $2^{\mathbb{N}}$.

Proof By Remark 1.1.3, any $B : \mathsf{Boole}$ is can be written as $C/(r_n)_{n:\mathbb{N}}$. By Remark 2.4.3, the quotient map induces an embedding $Sp(B) \hookrightarrow Sp(C) = 2^{\mathbb{N}}$, which is closed by by Theorem 3.3.1. □

(Can we maybe combine the next two Lemmas?)

Lemma 4.2.2 Assume $S : \mathsf{Stone}$ and $T : S \to \mathsf{Stone}$. Then $\Sigma_{x:S} T(x)$ is Compact Hausdorff.

Proof By Corollary 3.3.4 and Lemma 3.2.3, the identity types in $\Sigma_{x:S} T(x)$ are closed. By Lemma 4.2.1 we have for each $x : S$ that $\exists_{A:2^{\mathbb{N}} \to \mathsf{Closed}} T(x) = \Sigma_{y:2^{\mathbb{N}}} A(y)$. Using Axiom 3 we get $S' : \mathsf{Stone}$ with a surjective map: $q : S' \to S$ and: $C : S' \to (2^{\mathbb{N}} \to \mathsf{Closed})$ such that for all $x : S'$ we have $T(q(x)) = \Sigma_{y:2^{\mathbb{N}}} C(x, y)$. This gives a surjective map:

$$\Sigma_{c:(S' \times 2^{\mathbb{N}})} C(c) \twoheadrightarrow \Sigma_{x:S} T(x)$$

The source is Stone by Remark 3.1.4 and Remark 3.3.2 so we can conclude. □

Lemma 4.2.3 Assume $X : \mathsf{CHaus}$ and $T : X \to \mathsf{CHaus}$. Then $\Sigma_{x:X} T(x)$ is Compact Hausdorff.

Proof By Corollary 3.3.4 we have that identity type in $\Sigma_{x:X} T(x)$ are closed. We know that for any $x : X$ we have $\exists_{Y:\mathsf{Stone}} S' \twoheadrightarrow C(x)$. Consider the quotient map $q : S \twoheadrightarrow X$ with $S : \mathsf{Stone}$. By Axiom 3 we get $S' : \mathsf{Stone}$ with a surjective map: $e : S' \to S$ such that for all $x : S'$ we have $Y(x) : \mathsf{Stone}$ and a surjective map $Y(x) \to T(q(e(x)))$. This gives a surjective map:

$$\Sigma_{x:S'} Y(x) \to \Sigma_{x:X} T(x)$$

By Lemma 4.2.2 we have a surjective map from a Stone space to the source so we can conclude. □

4.3 Stone spaces are stable under dependent sums

We will show that Stone spaces are precisely totally disconnected compact Hausdorff spaces. We will use this to prove that a dependent sum of Stone spaces is Stone.

Lemma 4.3.1 Assume X compact Hausdorff, then 2^X is countably presented.

Proof Consider some quotient map $q : S \twoheadrightarrow X$ with $S : \mathsf{Stone}$. This induces an injection of Boolean algebras $2^X \hookrightarrow 2^S$. Note that $a : S \to 2$ lies in 2^X iff

$$\forall_{s,t:S} \left((q(s) =_X q(t)) \to (a(s) =_2 a(t)) \right).$$

As equality in X is closed and equality in 2 is decidable, Lemma 1.5.8 tells us that the implication is open for every $s, t : S$. By Corollary 4.1.3, we conclude that 2^X is an open subalgebra of 2^S. Therefore, it is in ODisc by Corollary 2.3.3 and Corollary 2.2.3 and in Boole by Corollary 2.4.2. □

Definition 4.3.2 Let X : CHaus and $x : X$. We define the connected component of x (denoted Q_x) as the intersection of all decidable subsets of X containing x.

Lemma 4.3.3 For all X : CHaus with $x : X$, Q_x is a countable intersection of decidables in X.

Proof By Lemma 4.3.1, we can enumerate the elements of 2^X, say as $(D_n)_{n:\mathbb{N}}$. Define E_n for $n : \mathbb{N}$ as D_n if $x \in D_n$ and X otherwise. Then $\cap_{n:\mathbb{N}} E_n = Q_x$. □

Lemma 4.3.4 Let X : CHaus, $x : X$ and suppose $U \subseteq X$ is open with $Q_x \subseteq U$. Then we have some decidable $E \subseteq X$ with $E(x)$ and $E \subseteq U$.

Proof By Lemma 4.3.3, we have $Q_x = \bigcap_{n:\mathbb{N}} D_n$ with $D_n \subseteq X$ decidable. If $Q_x \subseteq U$, we have that

$$Q_x \cap \neg U = \bigcap_{n:\mathbb{N}} (D_n \cap \neg U) = \emptyset.$$

By Lemma 4.1.4 there is some $N : \mathbb{N}$ with

$$(\bigcap_{n \leq N} D_n) \cap \neg U = \bigcap_{n \leq N} (D_n \cap \neg U) = \emptyset.$$

Therefore $\bigcap_{n \leq N} D_n \subseteq \neg\neg U$, which equals U by Remark 1.5.2. Note that decidable subsets are closed under finite intersection. Finally as $x \in D_n$ for all $n : \mathbb{N}$, $x \in \bigcap_{n \leq N} D_n$ as well and we're done. □

(We should define what connected means)

Lemma 4.3.5 Let X be Compact Hausdorff with $x : X$. Then Q_x is connected.

Proof Assume given a separation $Q_x = A \cup B$ with A, B disjoint and decidable in Q_x. Assume $x \in A$. By Lemma 4.3.3, $Q_x \subseteq X$ is closed. Using Remark 3.3.5, it follows that $A, B \subseteq X$ are closed and disjoint. By Lemma 4.1.8 there exist U, V disjoint open such that $A \subseteq U$ and $B \subseteq V$. By Lemma 4.3.4 we have a decidable D such that $Q_x \subseteq D \subseteq U \cup V$. Note that $D \cap U = D \cap (\neg V) := E$ is clopen, hence decidable by Corollary 1.5.5. Remark that $x \in E$, hence $B \subseteq Q_x \subseteq E$ but $B \cap E = \emptyset$, hence $B = \emptyset$. □

Lemma 4.3.6 Let X : CHaus, then X is Stone iff $\forall_{x:X} Q_x = \{x\}$.

Proof By Axiom 1, it is clear that for all $x : S$ with S Stone we have that $Q_x = \{x\}$. Conversely, assume that X : CHaus, $x : X$ and $Q_x = \{x\}$. We claim that the evaluation map $e : X \to Sp(2^X)$ is both injective and surjective, hence an equivalence. Let $x, y : X$. If $fx = fy$ for all $f : 2^X$, then $y \in Q_x$, hence $x = y$ by assumption. Thus e is injective. Let $q : S \twoheadrightarrow X$ be a quotient map. This induces an injection $2^X \hookrightarrow 2^S$, which by Axiom 2 induces a surjection $Sp(2^S) \twoheadrightarrow Sp(2^X)$. Note that $e \circ q$ factors as $S \simeq Sp(2^S) \twoheadrightarrow Sp(2^X)$. It follows that e is surjective. □

Theorem 4.3.7
Assume S : Stone and $T : S \to$ Stone. Then $\Sigma_{x:S} T(x)$ is Stone.

Proof By Lemma 4.2.2 we have that $\Sigma_{x:S} T(x)$ is compact Hausdorff. By Lemma 4.3.6 it is enough to show that for all $x : S$ and $y : T(x)$ we have that $Q_{(x,y)}$ is a singleton. Assume $(x', y') \in Q_{(x,y)}$, then for any map $f : S \to 2$ we have that:

$$f(x) = f \circ \pi_1(x, y) = f \circ \pi_1(x', y') = f(x')$$

so that $x' \in Q_x$ and since S is Stone by Lemma 4.3.6 we have that $x = x'$. Therefore we have $Q_{(x,y)} \subseteq \{x\} \times T(x)$. By Lemma 4.3.5, $Q_{(x,y)}$ is an inhabited connected subtype of a Stone space. Thus any map $T_x \to 2$ is constant on $Q_{(x,y)}$ and by Lemma 4.3.6 we conclude that it is a singleton. □

5 The Unit interval

5.1 The unit interval as Compact Hausdorff space

In this section we will introduce the unit interval I as compact Hausdorff space. The definition is based on [BB85].

Example 5.1.1 Let $n : \mathbb{N}$, we denote $C_n = 2[n]$ for the free Boolean algebra on n generators and no relations. Note that $Sp(C_n) = 2^n$ corresponds to the space of finite binary sequences.

Now we introduce some notation:

Definition 5.1.2

Given an infinite binary sequence $\alpha : 2^{\mathbb{N}}$ and a natural number $n : \mathbb{N}$ we denote $\alpha|_n : 2^n$ for the restriction of α to a finite sequence of length n.

We denote $\overline{0}, \overline{1}$ for the binary sequences which are constantly 0 and 1 respectively.

We denote $0, 1$ for the sequences of length 1 hitting $0, 1$ respectively.

If x is a finite sequence and y is any sequence, denote $x \cdot y$ for their concatenation.

Now we'll give a definition for when two finite binary sequences of length n correspond to real numbers whose distance is $\leq (\frac{1}{2})^n$. Informally, we want for every finite sequence s that $(s \cdot 0 \cdot \overline{1})$ and $(s \cdot 1 \cdot \overline{0})$ are equivalent.

Definition 5.1.3 Let $n : \mathbb{N}$ and let $s, t : 2^n$. We say s, t are n-near, and write $s \sim_n t$ if there merely exists some $m : \mathbb{N}$ and some $u : 2^m$, such that

$$\big((s = (u \cdot 0 \cdot \overline{1})|_n) \vee (s = (u \cdot 1 \cdot \overline{0})|_n)\big) \wedge \big((t = (u \cdot 0 \cdot \overline{1})|_n) \vee (t = (u \cdot 1 \cdot \overline{0})|_n)\big) \tag{16}$$

Remark 5.1.4

As we're dealing with finite sequences, $s \sim_n t$ is decidable.

Given any $s : 2^n$, using $m = n, u = s$ above, we can show that $s \sim_n s$. So n-nearness is reflexive.

Equation (16) is symmetric in s and t. Hence n-nearness is symmetric.

Note that $0 \cdot 0 \sim_2 0 \cdot 1 \sim_2 1 \cdot 0 \sim_2 1 \cdot 1$, but $0 \cdot 0 \nsim_2 1 \cdot 1$. Thus n-nearness is not in general transitive.

Definition 5.1.5 Let $\alpha, \beta : 2^{\mathbb{N}}$, we define $a \sim_I \beta$ as $\forall_{n:\mathbb{N}}(\alpha|_n \sim_n \beta|_n)$.

Lemma 5.1.6 Whenever $\alpha, \beta, \gamma : 2^{\mathbb{N}}$, are such that $\alpha \sim_I \beta, \beta \sim_I \gamma$, at least two of α, β, γ are equal.

Proof We will show that $\beta = \gamma \vee \alpha = \gamma \vee \alpha = \beta$. By Lemma 3.2.3 and Lemma 1.5.6, this is a closed proposition. By Remark 1.5.2, we can instead show the double negation. To this end, assume that none of α, β, γ are equal. By Theorem 1.4.2, there exist indices $i, j, k \in \mathbb{N}$ with

$$\beta(i) \neq \gamma(i), \alpha(j) \neq \gamma(j), \alpha(k) \neq \beta(k) \tag{17}$$

Let $n := \max(i, j, k) + 2$. As $\alpha \sim_I \beta$, we have $\alpha|_n \sim_n \beta|_n$. By assumption $\alpha|_n \neq \beta|_n$, so WLOG we may assume that we have some $m : \mathbb{N}, u : 2^m$ with

$$\alpha|_n = (u \cdot 0 \cdot \overline{1})|_n, \beta|_n = (u \cdot 1 \cdot \overline{0})|_n. \tag{18}$$

As $\alpha(k) \neq \beta(k)$ and $n \geq k + 2$, it follows in particular that $m \leq n - 2$ and hence $\beta(n-1) = 0$.
As also $\beta \sim_I \gamma$, we have $\beta|_n \sim_n \gamma|_n$. So there exists some $m' : \mathbb{N}, u' : 2^m$ with

$$\big((\beta|_n = (u' \cdot 0 \cdot \overline{1})|_n) \vee (\beta|_n = (u' \cdot 1 \cdot \overline{0})|_n)\big) \wedge \big((\gamma|_n = (u' \cdot 0 \cdot \overline{1})|_n) \vee (\gamma|_n = (u' \cdot 1 \cdot \overline{0})|_n)\big). \tag{19}$$

Similarly as above, we have that $m' \leq n - 2$, and as $\beta(n-1) = 0$, it follows that $\beta|_n = (u' \cdot 1 \cdot \overline{0})|_n$. Now as $\beta(i) \neq \gamma(i)$ with $i < n$, we have that $\beta|_n \neq \gamma|_n$, hence $\gamma|_n = (u' \cdot 0 \cdot \overline{1})|_n$. Now we have $m, m' \leq n - 2$ and $u : 2^m, u' : 2^{m'}$ such that

$$(u \cdot 1 \cdot \overline{0})|_n = \beta|_n = (u' \cdot 1 \cdot \overline{0})|_n \tag{20}$$

Note that $\beta(m') = 1$. But also $\beta(l) = 0$ for all l with $m < l < n$ Therefore $m' \leq m$. By similar reasoning, $m \leq m'$. We conclude $m = m'$. As a consequence, $u = u'$, but then $\gamma|_n = \alpha|_n$, contradicting that $\alpha(j) \neq \gamma(j)$ for $j < n$. Hence we arrive at a contradiction, as required. ☐

Corollary 5.1.7 $\sim_I$ is a closed equivalence relation on $2^{\mathbb{N}}$.

Proof By Remark 5.1.4, $\sim_I$ is a countable conjunction of decidable propositions. Also by Remark 5.1.4, $\sim_n$ is reflexive and symmetric for all $n : \mathbb{N}$, thus $\sim_I$ is reflexive and symmetric as well. Finally $\sim_I$ is transitive as a consequence of Lemma 5.1.6. ☐

Definition 5.1.8 We define $\mathbb{I} : \mathsf{CHaus}$ as $\mathbb{I} = 2^{\mathbb{N}} / \sim_I$.

5.2 Order on the interval

Definition 5.2.1 For $n : \mathbb{N}$ we define $cs_n : 2^n \to \mathbb{Q}$ by

$$cs_n(a) = \sum_{i=0}^{n-1} \frac{a(i)}{2^{i+1}} \tag{21}$$

And for $\alpha : 2^{\mathbb{N}}$, we define the sequence $cs(\alpha) : \mathbb{N} \to \mathbb{Q}$ by

$$cs(\alpha)_n = cs_n(\alpha|_n) \tag{22}$$

Remark 5.2.2 cs_n gives a bijection between 2^n and it's image $\{\frac{k}{2^n} | 0 \leq k \leq 2^n - 1\} \subseteq \mathbb{Q}$. This observation has some corollaries:

- In particular, each cs_n is injective.
- Furthermore, whenever $a \neq b : 2^n$, we must have that

$$|cs_n(a) - cs_n(b)| \geq \frac{1}{2^n}. \tag{23}$$

- It is known that $\bigcup_{n:\mathbb{N}} \{\frac{k}{2^n} | 0 \leq k \leq 2^n - 1\}$ lies dense in the interval of Cauchy reals $[0, 1]$. It follows that cs induces a surjection from Cantor space to $[0, 1]$. We claim without proof it in fact induces an equivalence between $\mathbb{I}$ and $[0, 1]$.

Finally, let us repeat a well-known identity for all $m < n$ on such sums, which we'll make some use of

$$\sum_{i=m}^{n-1} \frac{1}{2^{i+1}} = \frac{1}{2^m} - \frac{1}{2^n} \tag{24}$$

Lemma 5.2.3 Let $n : \mathbb{N}$ and $s, t : 2^n$. Assume there is some $m \leq n$ with $cs_m(s|_m) = cs_m(t|_m) + \frac{1}{2^m}$, and at the same time $cs_n(s) - cs_n(t) \leq \frac{1}{2^n}$. Then there is some $k < m$ and some $u : 2^k$ such that

$$(s = u \cdot 1 \cdot \overline{0}|_n) \wedge (t = u \cdot 0 \cdot \overline{1}|_n) \tag{25}$$

Proof By assumption, we have that $s|_m \neq t|_m$. Then there must be some smallest number $k < m$ such that $s(k) \neq t(k)$. As k is minimal, we have $s|_k = t|_k =: u$. It follows for all $l \leq n$ that

$$cs_l(s|_l) - cs_l(t|_l) = \sum_{i=k}^{l-1} \frac{s(i) - t(i)}{2^{i+1}} \tag{26}$$

Note that as $s(i), t(i) \in \{0, 1\}$, we must have $|s(i) - t(i)| \leq 1$. Hence for any $k' < l$, we have

$$\left| \sum_{i=k'}^{l-1} \frac{s(i) - t(i)}{2^{i+1}} \right| \leq \sum_{i=k'}^{l-1} \frac{1}{2^{i+1}} = \frac{1}{2^{k'}} - \frac{1}{2^l} \tag{27}$$

Note that using the two equations above for $l = m$ and $k' = k + 1$ we have:

$$cs_m(s|_m) - cs_m(t|_m) = \frac{s(k) - t(k)}{2^{k+1}} + \sum_{i=k+1}^{m-1} \frac{s(i) - t(i)}{2^{i+1}} \tag{28}$$

$$\leq \frac{s(k) - t(k)}{2^{k+1}} + \left(\frac{1}{2^{k+1}} - \frac{1}{2^m} \right) \tag{29}$$

As the left hand side should equal $\frac{1}{2^m}$, we must have that $s(k) - t(k) \neq -1$. As $s(k) \neq t(k)$ it follows that $s(k) = 1, t(k) = 0$. But now

$$cs_n(s) - cs_n(t) = \frac{1}{2^{k+1}} + \sum_{i=k+1}^{n-1} \frac{s(i) - t(i)}{2^{i+1}} \geq \frac{1}{2^{k+1}} - \left(\frac{1}{2^{k+1}} - \frac{1}{2^n} \right) = \frac{1}{2^n} \tag{30}$$

And as $cs_n(s) - cs_n(t) \leq \frac{1}{2^n}$ as well, we get that $cs_n(s) - cs_n(t) = \frac{1}{2^n}$. Note that this lower bound is only reached if $s(i) - t(i) = -1$ for all $k < i < n$. Hence for those i, we have $s(i) = 0, t(i) = 1$. Thus

$$s = (u \cdot 1 \cdot \overline{0})|_n \wedge t = (u \cdot 0 \cdot \overline{1})|_n. \tag{31}$$

□

Corollary 5.2.4 Let $n : \mathbb{N}$ and let $s, t : 2^n$. Then

$$s \sim_n t \leftrightarrow |cs_n(s) - cs_n(t)| \leq \frac{1}{2^n}. \tag{32}$$

Proof

Assume $s \sim_n t$. If $s = t$, we have $cs_n(s) - cs_n(t) = 0$, otherwise, we may without loss of generality assume there is some $m < n$ and some $u : 2^m$ such that

$$(s = u \cdot 0 \cdot \overline{1}|_n) \wedge (t = u \cdot 1 \cdot \overline{0}|_n). \tag{33}$$

Then

$$cs_n(s) = cs_m(u) + 0 + \sum_{i=m+1}^{n-1} \frac{1}{2^{i+1}} \tag{34}$$

$$cs_n(t) = cs_m(u) + \frac{1}{2^{m+1}} + 0 \tag{35}$$

And hence

$$cs_n(t) - cs_n(s) = \frac{1}{2^{m+1}} - \sum_{i=m+1}^{n-1} \frac{1}{2^{i+1}} = \frac{1}{2^n} \tag{36}$$

Thus in all cases, from $s \sim_n t$, we can conclude that

$$|cs_n(s) - cs_n(t)| \leq \frac{1}{2^n} \tag{37}$$

Conversely, assume that $|cs_n(s) - cs_n(t)| \leq \frac{1}{2^n}$. If $s = t$, it is clear that $s \sim_n t$. If $s \neq t$, there must be some smallest number $m < n$ such that $s(m) \neq t(m)$. As m is minimal, we have $s|_m = t|_m =: u$. WLOG, we assume that $s(m) = 1, t(m) = 0$. Then $cs_m(s|_{m+1}) = cs_{m+1}(t|_{m+1}) + \frac{1}{2^{m+1}}$ and by Lemma 5.2.3 it follows that

$$s = (u \cdot 1 \cdot \overline{0})|_n \wedge t = (u \cdot 0 \cdot \overline{1})|_n. \tag{38}$$

and thus we can conclude $s \sim_n t$ as required. □

Inspired by Definitions 2.7 and 2.10 [BB85], we define inequality on $\mathbb{I}$ as follows:

Definition 5.2.5 Let $\alpha, \beta : 2^{\mathbb{N}}$. We define $\alpha \leq_{\mathbb{I}} \beta$ and $\alpha <_{\mathbb{I}} \beta$ as follows:

$$\alpha \leq_{\mathbb{I}} \beta := \forall_{n:\mathbb{N}} \left(cs(\alpha)_n \leq cs(\beta)_n + \frac{1}{2^n} \right) \tag{39}$$

$$\alpha <_{\mathbb{I}} \beta := \exists_{n:\mathbb{N}} \left(cs(\alpha)_n < cs(\beta)_n - \frac{1}{2^n} \right) \tag{40}$$

Lemma 5.2.6 $\leq_{\mathbb{I}}$ respects $\sim_{\mathbb{I}}$.

Proof We will show that whenever $\alpha \leq_{\mathbb{I}} \gamma$ and $\alpha \sim_{\mathbb{I}} \beta$, we have $\beta \leq_{\mathbb{I}} \gamma$. The other proof obligation goes similarly.

As $\beta \leq_{\mathbb{I}} \gamma$ is closed, by Remark 1.5.2 it is double negation stable. By Theorem 1.4.2, the negation is that there is some $N : \mathbb{N}$ with $cs(\beta)_N > cs(\gamma)_N + \frac{1}{2^N}$. As $\alpha \leq_{\mathbb{I}} \gamma$, we have $cs(\gamma)_N + \frac{1}{2^N} \geq cs(\alpha)_N$. Thus $cs(\beta)_N > cs(\alpha)_N$ and therefore $cs(\beta)_N = cs(\alpha)_N + \frac{1}{2^N}$ using $\alpha \sim_{\mathbb{I}} \beta$. It follows that

$$cs(\alpha)_N + \frac{1}{2^N} > cs(\gamma)_N + \frac{1}{2^N} \geq cs(\alpha)_N$$

From Remark 5.2.2, we must have $cs(\gamma)_N + \frac{1}{2^N} = cs(\alpha)_N$, otherwise the distance between $cs(\gamma)_N$ and $cs(\alpha)_N$ would be smaller than $\frac{1}{2^N}$. As $cs(\alpha)_n \leq cs(\gamma)_n + \frac{1}{2^n}$ for all $n \geq N$, Lemma 5.2.3 gives that $\alpha \sim_{\mathbb{I}} \gamma$. But also $\beta \sim_{\mathbb{I}} \gamma$. But now α, β, γ are all distinct yet related by $\sim_{\mathbb{I}}$, contradicting Lemma 5.1.6.□

Remark 5.2.7 By Theorem 1.4.2, we have that $\neg(\alpha \leq \beta) \leftrightarrow (\beta <_{\mathbb{I}} \alpha)$. It follows immediately that $<_{\mathbb{I}}$ also respects $\mathbb{I}$. Therefore, $\leq_{\mathbb{I}}, <_{\mathbb{I}}$ induce relations $\leq, <$ on $\mathbb{I}$. As the order in $\mathbb{Q}$ is decidable, $\leq, <$ are closed and open respectively.

Lemma 5.2.8 For any $x, y : \mathbb{I}$, we have $x \leq y \vee y \leq x$.

Proof Note that $x \leq y \vee y \leq x$ is the disjunction of two closed propositions, hence by Lemma 1.5.6 and Remark 1.5.2 we can show it's double negation instead. By the above remark, the negation implies that $x > y$ and $y < x$. We will show this is a contradiction. Let $\alpha, \beta : 2^{\mathbb{N}}$ correspond to x, y and assume $n, m : \mathbb{N}$ with $cs(\alpha)_n < cs(\beta)_n - \frac{1}{2^n}$ and $cs(\beta)_m < cs(\alpha)_m - \frac{1}{2^m}$. WLOG assume $n < m$. In this case for γ any of α, β, we have

$$0 \leq cs(\gamma)_m - cs(\gamma)_n = \sum_{i=n}^{m-1} \frac{\gamma(i)}{2^{i+1}} \leq \frac{1}{2^n} - \frac{1}{2^m}$$

While at the same time, we have

$$\begin{aligned} cs(\beta)_m - cs(\beta)_n &\leq cs(\alpha)_m - \frac{1}{2^m} - cs(\beta)_n && (41) \\ &= (cs(\alpha)_m - cs(\alpha)_n) + (cs(\alpha)_n - cs(\beta)_n) - \frac{1}{2^m} && (42) \\ &\leq (\frac{1}{2^n} - \frac{1}{2^m}) - \frac{1}{2^n} - \frac{1}{2^m} && (43) \\ &< 0 && (44) \end{aligned}$$

giving a contradiction as required. □

Remark 5.2.9 From Corollary 5.2.4 we have $((x \leq y) \wedge (y \leq x)) \leftrightarrow (x = y)$. So in order to define a map $(x \leq y) \vee (y \leq x) \to P$, we need to define a map $f : x \leq y \to P$ and a map $g : y \leq x \to P$ such that $f|_{x=y} = g|_{x=y}$.

(These properties are nice but not necessary and paused WIP:) (It is no used for Bouwer's fixed point theorem)

Corollary 5.2.10 For $x, y : \mathbb{I}$ we have $(x \leq y \wedge x \neq y) \leftrightarrow (x < y)$. Also $(x \neq y) \leftrightarrow (x < y + x > y)$.

Proof By $(x < y) \leftrightarrow \neg(y \leq x)$ It's also immediate from the definitions that $x < y$ implies $x \neq y$. As $((x \leq y) \wedge (y \leq x)) \leftrightarrow (x = y)$, if $x \leq y \wedge x \neq y$, we have $\neg(y \leq x)$, hence $x < y$. □

Lemma 5.2.11 Whenever $x, y : \mathbb{I}$ satisfy $x < y$, there is some $z : \mathbb{I}$ with $x < z \wedge z < y$.

5.3 The topology of the interval

Definition 5.3.1 Let $a, b : \mathbb{I}$. Following standard notation, we denote

$$[a, b] := \Sigma_{x:\mathbb{I}}(a \leq x \wedge x \leq b), \qquad (45)$$

we call subsets of $\mathbb{I}$ of this form closed intervals. We also denote

$$\begin{aligned} (-\infty, a) &:= \Sigma_{x:\mathbb{I}}(x < a) && (46) \\ (a, \infty) &:= \Sigma_{x:\mathbb{I}}(a < x) && (47) \\ (-\infty, \infty) &:= \mathbb{I} && (48) \\ (a, b) &:= \Sigma_{x:\mathbb{I}}(a < x \wedge x < b), && (49) \end{aligned}$$

We call subsets of $\mathbb{I}$ of these forms open intervals.

Remark 5.3.2 Note that closed intervals and open intervals are closed and open respectively.

Lemma 5.3.3 For $p : 2^{\mathbb{N}} \to \mathbb{I}$ the quotient map and $D \subseteq 2^{\mathbb{N}}$ decidable, we have $p(D)$ a finite union of closed intervals.

Proof We will show the above if there exists some $n : \mathbb{N}, u : 2^n$ such that $D(\alpha) \leftrightarrow \alpha|_n = u$. This is sufficient, as any decidable subset of $2^{\mathbb{N}}$ can be written as finite union of such decidable subsets. We claim that $p(D) = [p(u \cdot \overline{0}), p(u \cdot \overline{1})]$.

We will first show that $p(D) \subseteq [p(u \cdot \overline{0}), p(u \cdot \overline{1})]$. Suppose $D(\alpha)$. Then Then $\alpha|_n = u$ and hence

$$cs(u \cdot \overline{1})_m \geq cs(\alpha)_m \geq cs(u \cdot \overline{0})_m \qquad (50)$$

which implies that $p(u \cdot \overline{1}) \geq_{\mathbb{I}} p(\alpha) \geq_{\mathbb{I}} p(u \cdot \overline{0})$, as required.

To show that $[p(u \cdot \overline{0}), p(u \cdot \overline{1})] \subseteq p(D)$, Suppose $(u \cdot \overline{0}) \leq_{\mathbb{I}} \alpha \leq_{\mathbb{I}} (u \cdot \overline{1})$. It is sufficient to show that

$$(\alpha|_n = u) \vee (\alpha \sim_{\mathbb{I}} u \cdot \overline{0}) \vee (\alpha \sim_{\mathbb{I}} u \cdot \overline{1}).$$

As this is a disjunction of closed propositions, by Lemma 1.5.6 it's closed, and by Remark 1.5.2, we can instead show the double negation. So suppose that none of the disjoints hold. As $\alpha|_n \neq u$, there is some minimal m with $\alpha(m) \neq u(m)$. We assume that $\alpha(m) = 1, u(m) = 0$, the other case goes similarly. Then for all $k : \mathbb{N}$, we have $cs(\alpha)_k \geq cs(u \cdot \overline{1})|_k$. As also $(u \cdot \overline{1}) \geq_{\mathbb{I}} \alpha$, we have

$$cs(u \cdot \overline{1})|_k + \frac{1}{2^k} \geq cs(\alpha)_k \geq cs(u \cdot \overline{1})_k,$$

From which it follows that $|cs(u \cdot \overline{1})_k - cs(\alpha)_k| \leq \frac{1}{2^k}$. Hence $(u \cdot \overline{1})|_k \sim_k \alpha|_k$ by Corollary 5.2.4. Hence $x \sim_{\mathbb{I}} (a \cdot \overline{1})$, contradicting our assumption as required. □

Lemma 5.3.4 The complement of a finite union of closed intervals is a finite union of open intervals.

Proof We'll use induction on the amount of closed intervals. The empty union of closed intervals is empty, and hence it's complement is $\mathbb{I}$, which is an open interval. Let $(C_i)_{i<k}$ be a finite set of closed intervals with $\neg(\bigcup_{i<k} C_i)$ a finite union of open intervals $\bigcup_{j<l} O_i$. Suppose C_k is closed. We need to show that $\neg(\bigcup_{i\leq k} C_i)$ is also a finite union of open intervals. First note that in general, $(\neg(A \vee B)) \leftrightarrow (\neg A \wedge \neg B)$ hence

$$\neg(\bigcup_{i\leq k} C_i) = \neg((\bigcup_{i<k} C_i) \cup C_k) = (\neg(\bigcup_{i<k} C_i)) \cap (\neg C_k)$$

And by the induction hypothesis and distributivity, this equals

$$(\bigcup_{j<l} O_i)) \cap (\neg C_k) = \bigcup_{j<l} (O_i \cap (\neg C_k))$$

So we need to show that the intersection of an open interval and the negation of a closed interval is a finite union of open intervals. We assume or open intervals are of the form (a, b) for $a, b : \mathbb{I}$. The other cases are very similar. So let $a, b, c, d : \mathbb{I}$ and consider $U = (a, b) \cap (\neg[c, d])$. Then

$$\begin{aligned}
U(x) &= \Sigma_{x:\mathbb{I}}(a < x \wedge x < b) \wedge (x < c \vee d < x) &(51)\\
&= \Sigma_{x:\mathbb{I}}(a < x \wedge x < b \wedge x < c) \vee (d < x \vee a < x \wedge x < b) &(52)\\
&= \Sigma_{x:\mathbb{I}}(a < x \wedge x < b \wedge x < c) \cup \Sigma_{x:\mathbb{I}}(d < x \vee a < x \wedge x < b) &(53)
\end{aligned}$$

We will show that $U' = \Sigma_{x:\mathbb{I}}(a < x \wedge x < b \wedge x < c)$ is an open interval. By a similar argument, the other part will be as well, meaning that U is the union of two open intervals. Consider that $b \leq c \vee c \leq b$. If $b \leq c$, $(x < b \wedge x < c) \leftrightarrow x < b$ and $U' = (a, b)$ If $c \leq b$, $(x < b \wedge x < c) \leftrightarrow x < c$ and $U' = (a, c)$ If $b = c$, these open intervals agree, hence from Remark 5.2.9 we can conclude that U' is an open interval. We conclude that U is the union of two open intervals as required. □

Lemma 5.3.5 Every open $U \subseteq \mathbb{I}$ can be written as countable union of open intervals.

Proof By Corollary 4.1.6 there is some sequence of decidable subsets $D_n \subseteq 2^{\mathbb{N}}$ with $U = \bigcup_{n:\mathbb{N}} \neg p(D_n)$. By Lemma 5.3.3, each $p(D_n)$ is a finite union of closed intervals, and by Lemma 5.3.4 it follows that each $\neg p(D_n)$ is a finite union of open intervals. We conclude that U is a countable union of open intervals as required. □

Remark 5.3.6 It follows that the topology of $\mathbb{I}$ is generated by open intervals, which corresponds to the standard topology on $\mathbb{I}$. Hence our notion of continuity corresponds with the ϵ, δ-definition of continuity one would expect. Thus every function $f : \mathbb{I} \to \mathbb{I}$ in the system we presented is continuous in the ϵ, δ-sense.

6 Cohomology

In this section we compute $H^1(S, \mathbb{Z}) = 0$ for S Stone, and show that $H^1(X, \mathbb{Z})$ for X compact Hausdorff can be computed using Čech cohomology. We then apply this to compute $H^1(\mathbb{I}, \mathbb{Z}) = 0$. We work on the first cohomology group with coefficient in $\mathbb{Z}$ as it is sufficient for the proof of Brouwer's fixed-point theorem, but the results could be extended to $H^n(X, A)$ for A any family of countably presented abelian groups indexed by X.

6.1 Needed results

(Should probably be moved elsewhere)

Lemma 6.1.1 Given S : Stone and $T : S \to$ Stone such that $\prod_{x:S}\|T(x)\|$, there exists a sequence of finite types $(S_k)_{k:\mathbb{N}}$ such that:

$$\lim_k S_k = S$$

and for each $k : \mathbb{N}$ we have a family of finite types $T_k(x)$ for $x : S_k$ such that $\prod_{x:S_k}\|T_k(x)\|$ with maps:

$$T_{k+1}(x) \to T_k(p_k(x))$$

such that:

$$\lim_k\left(\sum_{x:S_k} T_k(x)\right) = \sum_{x:S} T(x)$$

Proof (Hugo This follows from Remark 3.1.4 and Theorem 4.3.7 and considering the surjection $(\Sigma_{x:S}T(x)) \to S$, but we discussed whether it might be easier to refactor the proof where you use the above or make a remark after Theorem 4.3.7) □

Lemma 6.1.2 Consider $(S_k)_{k:\mathbb{N}}$ a sequence of finite types, then the canonical map:

$$\mathrm{colim}_k(\mathbb{Z}^{S_k}) \to \mathbb{Z}^{\lim_k S_k}$$

is an equivalence

Proof TODO □

6.2 Čech cohomology

Definition 6.2.1 Given S : Type with $T : S \to$ Type and $A : S \to$ Ab, we define $\check{C}(S, T, A)$ as the chain complex:

$$\prod_{x:S} A_x^{T_x} \xrightarrow{d_0} \prod_{x:S} A_x^{T_x^2} \xrightarrow{d_1} \prod_{x:S} A_x^{T_x^3} \tag{54}$$

with the usual boundary maps:

$$d_0(\alpha)_x(u, v) = \alpha_x(v) - \alpha_x(u)$$
$$d_1(\beta)_x(u, v, w) = \beta_x(v, w) - \beta_x(u, w) + \beta_x(u, w)$$

Definition 6.2.2 Given S : Type with $T : S \to$ Type and $A : S \to$ Ab, we define its Čech cohomology groups by:

$$\check{H}^0(S, T, A) = \ker(d_0)$$
$$\check{H}^1(S, T, A) = \ker(d_1)/\mathrm{im}(d_0)$$

This means that $\check{H}^1(S, T, A) = 0$ if and only if $\check{C}(S, T, A)$ is exact. Now we give three very general lemmas about Čech complexes.

Lemma 6.2.3 Given S : Type with $T : S \to$ Type and $A : S \to$ Ab with $t : \prod_{x:S} T_x$, then we have that $\check{C}(S, T, A)$ is exact.

Proof Assume given a cocycle, i.e. $\beta : \prod_{x:S} A_x^{T_x^2}$ such that for all $x : S$ and $u, v, w : T_x$ we have that:

$$\beta_x(u, v) + \beta_x(v, w) = \beta_x(u, w)$$

Define:

$$\alpha : \prod_{x:S} A_x^{T_x}$$
$$\alpha_x(u) = \beta_x(t_x, u)$$

Then for all $x : S$ and $u, v : T_x$ we have that:

$$\alpha_x(v) - \alpha_x(u) = \beta_x(t_x, v) - \beta_x(t_x, u) = \beta_x(u, t_x) + \beta_x(t_x, v) = \beta_x(u, v)$$

so that β is a coboundary. □

Lemma 6.2.4 Given S : Type with $T : S \to$ Type and $A : S \to$ Ab, we have that $\check{C}(S, T, \lambda x.A_x^{T_x})$ is exact.

Proof Assume given a cocycle, i.e. $\beta : \prod_{x:S} A_x^{T_x^3}$ such that for all $x : S$ and $u, v, w, t : T_x$ we have that:

$$\beta_x(u, v, t) + \beta_x(v, w, t) = \beta_x(u, w, t)$$

Define:

$$\alpha : \prod_{x:S} A_x^{T_x^2}$$

$$\alpha_x(u, t) = \beta_x(t, u, t)$$

Then for all $x : S$ and $u, v, t : T_x$ we have that:

$$\alpha_x(v, t) - \alpha_x(u, t) = \beta_x(t, v, t) - \beta_x(t, u, t) = \beta_x(u, t, t) + \beta_x(t, v, t) = \beta_x(u, v, t)$$

so that β is a coboundary. $\square$

Lemma 6.2.5 Given S : Type with $T : S \to$ Type and $A : S \to$ Ab, assume that $\check{C}(S, T, A)$ is exact. Then given:

$$\alpha : \prod_{x:S} BA_x$$

with

$$\beta : \prod_{x:S} (\alpha(x) = *)^{T_x}$$

we can conclude that:

$$\alpha = *$$

Proof We define:

$$g : \prod_{x:S} A_x^{T_x^2}$$

$$g_x(u, v) = \beta_x(u)^{-1} \cdot \beta_x(v)$$

It is a cocycle in the Čech complex, so that by exactness there is $f : \prod_{x:S} A_x^{T_x}$ such that for all $x : S$ and $u, v : T_x$ we have that:

$$g_x(u, v) = f_x(v) \cdot f_x(u)^{-1}$$

Then we define:

$$\beta' : \prod_{x:S} (\alpha(x) = *)^{T_x}$$

$$\beta'_x(u) = \beta_x(u) \cdot f_x(u)$$

so that for all $x : S$ and $u, v : T(x)$ we have that:

$$\beta'_x(v)^{-1} \cdot \beta'_x(u) = f_x(v)^{-1} \cdot \beta_x(v)^{-1} \cdot \beta_x(u) \cdot f_x(u) = \mathrm{refl}$$

so that:

$$\beta'_x(u) = \beta'_x(v)$$

and this means that β' factors through S, giving a proof of $\alpha = *$. $\square$

6.3 Cohomology of Stone spaces

Lemma 6.3.1 Given S : Stone with $T : S \to \mathsf{Stone}$, if $\prod_{x:S} \|T_x\|$ then we have that $\check{C}(S, T, \mathbb{Z})$ is exact.

Proof We apply lemma 6.1.1 to get S_k and T_k finite. Then by lemma 6.1.2 we have that:

$$\mathrm{colim}_k \check{C}(S_k, T_k, \mathbb{Z}) = \check{C}(S, T, \mathbb{Z})$$

and each of the $\check{C}(S_k, T_k, \mathbb{Z})$ is exact by lemma 6.2.3 so we can conclude since a sequential colimit of exact sequence is exact. $\square$

Lemma 6.3.2 Given S : Stone, we have that $H^1(S, \mathbb{Z}) = 0$.

Proof Assume given a map $\alpha : S \to B\mathbb{Z}$. We use local choice to get $T : S \to \mathsf{Stone}$ such that $\prod_{x:S} \|T_x\|$ and:

$$\beta : \prod_{x:S} (\alpha(x) = *)^{T_x}$$

and then apply lemma 6.3.1 and lemma 6.2.5 to conclude. $\square$

Corollary 6.3.3 For any $S : \mathsf{Stone}$ the canonical map:

$$B(\mathbb{Z}^S) \to B\mathbb{Z}^S$$

is an equivalence.

6.4 Čech cohomology of compact Hausdorff spaces

Definition 6.4.1 A Čech cover consists of $X : \mathsf{CHaus}$ and $S : X \to \mathsf{Stone}$ such that $\prod_{x:X} \|S_x\|$ and $\sum_{x:X} S_x : \mathsf{Stone}$.

By definition any compact Hausdorff type has a Čech cover.

Lemma 6.4.2 Given a Čech cover (X, S), we have that:

$$H^0(X, \mathbb{Z}) = \check{H}^0(X, S, \mathbb{Z})$$

Proof By definition an element in $\check{H}^0(X, S, \mathbb{Z})$ is a map:

$$f : \prod_{x:X} \mathbb{Z}^{S_x}$$

such that for all $u, v : S_x$ we have $f(u) = f(v)$, which is equivalent to a map:

$$\prod_{x:X} \mathbb{Z}^{\|S_x\|}$$

since $\mathbb{Z}$ is set, and since the S_x are merely inhabited this is the same as $\mathbb{Z}^X$. $\square$

Lemma 6.4.3 Given a Čech cover (X, S) we have an exact sequence:

$$H^0(X, \lambda x.\mathbb{Z}^{S_x}) \to H^0(X, \lambda x.\mathbb{Z}^{S_x}/\mathbb{Z}) \to H^1(X, \mathbb{Z}) \to 0$$

Proof We use the long exact cohomology sequence associated to:

$$0 \to \mathbb{Z} \to \mathbb{Z}^{S_x} \to \mathbb{Z}^{S_x}/\mathbb{Z} \to 0$$

so that we just need $H^1(X, \lambda x.\mathbb{Z}^{S_x}) = 0$ to conclude. But by corollary 6.3.3 we have that:

$$H^1(X, \lambda x.\mathbb{Z}^{S_x}) = H^1\left(\sum_{x:X} S_x, \mathbb{Z}\right)$$

which vanishes by lemma 6.3.2. $\square$

Lemma 6.4.4 Given a Čech cover (X, S) we have an exact sequence:

$$\check{H}^0(X, \lambda x.\mathbb{Z}^{S_x}) \to \check{H}^0(X, \lambda x.\mathbb{Z}^{S_x}/\mathbb{Z}) \to \check{H}^1(X, \mathbb{Z}) \to 0$$

Proof By lemma 6.3.2 we have an exact sequence of complexes:

$$0 \to \check{C}(X, S, \mathbb{Z}) \to \check{C}(X, S, \lambda x.\mathbb{Z}^{S_x}) \to \check{C}(X, S, \lambda x.\mathbb{Z}^{S_x}/\mathbb{Z})$$

But since $\check{H}^1(X, \lambda x.\mathbb{Z}^{S_x}) = 0$ by lemma 6.2.4, we conclude using the associated long exact sequence. $\square$

Theorem 6.4.5
Given a Čech cover (X, S), we have that:

$$H^1(X, \mathbb{Z}) = \check{H}^1(X, S, \mathbb{Z})$$

Proof We apply lemma 6.4.2, lemma 6.4.3 and lemma 6.4.4. $\square$

In particular this means that the Čech cohomology $\check{H}^1(X, S, \mathbb{Z})$ do not depend on S.

6.5 Cohomology of the interval

Recall that we denote $C_n = 2^n$ with a binary relation $\sim_n$ on C_n such that for all $x, y : 2^{\mathbb{N}}$ we have that:

$$(\forall (n : \mathbb{N}).\ x|_n \sim_n y|_n) \leftrightarrow x =_{\mathbb{I}} y$$

Lemma 6.5.1 We have that $(C_n, \sim_n)$ is equivalent to $(\mathrm{Fin}(2^n), \lambda x, y.\ |x - y| \leq 1)$.

Proof By corollary 5.2.4. $\square$

We write:

$$C_n^{\sim 2} = \sum_{x,y:C_n} x \sim_n y$$

$$C_n^{\sim 3} = \sum_{x,y,z:C_n} x \sim_n y \wedge y \sim_n z \wedge x \sim_n z$$

Lemma 6.5.2 For any $n : \mathbb{N}$ we have an exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z}^{C_n} \to \mathbb{Z}^{C_n^{\sim 2}} \to \mathbb{Z}^{C_n^{\sim 3}}$$

with the obvious boundary maps.

Proof It is clear that the map $\mathbb{Z} \to \mathbb{Z}^{C_n}$ is injective as C_n is inhabited, so the sequence is exact at $\mathbb{Z}$.

Assume a cocycle $\alpha : \mathbb{Z}^{C_n}$, meaning that for all $u, v : C_n$, if $u \sim_n v$ then $\alpha(u) = \alpha(v)$. Then by lemma 6.5.1 we see that α is constant, so the sequence is exact at $\mathbb{Z}^{C_n}$.

Assume a cocycle $\beta : \mathbb{Z}^{C_n^{\sim 2}}$, meaning that for all $u, v, w : C_n$ such that $u \sim_n v$, $v \sim_n w$ and $u \sim_n w$ we have that:

$$\beta(u, v) + \beta(v, w) = \beta(u, w)$$

which is equivalent to asking $\beta(u, u) = 0$ and $\beta(u, v) = -\beta(v, u)$.

Using lemma 6.5.1 we can define:

$$\alpha(n) = \beta(0, 1) + \cdots + \beta(n - 1, n)$$

Then for all (m, n) such that $|m - n| \leq 1$ we have that:

- If $m = n$ and then $\beta(m, m) = 0 = \alpha(m) - \alpha(m)$.
- If $m + 1 = n$ then $\beta(m, m + 1) = \alpha(m + 1) - \alpha(m)$.
- If $m = n + 1$ then $\beta(n + 1, n) = -\beta(n, n + 1) = -\alpha(n + 1) + \alpha(n)$.

So β is indeed a coboundary and the sequence is exact at $\mathbb{Z}^{C_n^{\sim 2}}$. $\square$

Proposition 6.5.3 We have that:

$$H^0(\mathbb{I}, \mathbb{Z}) = \mathbb{Z}$$
$$H^1(\mathbb{I}, \mathbb{Z}) = 0$$

Proof Consider the canonical surjective map $p : 2^{\mathbb{N}} \to \mathbb{I}$ and the associated Čech cover of $\mathbb{I}$ by:

$$T_x = \sum_{y:2^{\mathbb{N}}} x =_{\mathbb{I}} p(y)$$

Then for $l = 2, 3$ we have that:

$$\lim_n C_n^{\sim l} = \sum_{x:\mathbb{I}} T_x^l$$

By lemma 6.5.2 and stability of exactness under sequential colimit, we know that:

$$0 \to \mathbb{Z} \to \mathrm{colim}_n \left(\mathbb{Z}^{C_n}\right) \to \mathrm{colim}_n \left(\mathbb{Z}^{C_n^{\sim 2}}\right) \to \mathrm{colim}_n \left(\mathbb{Z}^{C_n^{\sim 3}}\right)$$

is exact, but by lemma 6.1.2 this sequence is equivalent to:

$$0 \to \mathbb{Z} \to \prod_{x:\mathbb{I}} \mathbb{Z}^{T_x} \to \prod_{x:\mathbb{I}} \mathbb{Z}^{T_x^2} \to \prod_{x:\mathbb{I}} \mathbb{Z}^{T_x^3}$$

So it being exact implies that:

$$\check{H}^0(\mathbb{I}, T, \mathbb{Z}) = \mathbb{Z}$$
$$\check{H}^1(\mathbb{I}, T, \mathbb{Z}) = 0$$

We conclude by lemma 6.4.2 and theorem 6.4.5. $\square$

Remark 6.5.4 We could carry a similar computation for $\mathbb{S}^1$, by approximating it with 2^n with $0^n \sim_n 1^n$ added. We would find $H^1(\mathbb{S}^1, \mathbb{Z}) = \mathbb{Z}$.

6.6 Brouwer's fixed-point theorem

Here we consider the modality defined by localising at $\mathbb{I}$ [RSS20], denoted by $L_{\mathbb{I}}$. We say X is $\mathbb{I}$-local if $L_{\mathbb{I}}(X) = X$ and that it is $\mathbb{I}$-contractible if $L_{\mathbb{I}}(X) = 1$.

Lemma 6.6.1 We have that $B\mathbb{Z}$ is $\mathbb{I}$-local.

Proof By proposition 6.5.3, from $H^0(\mathbb{I}, \mathbb{Z}) = \mathbb{Z}$ we get that the map $\mathbb{Z} \to \mathbb{Z}^{\mathbb{I}}$ is an equivalence, so $\mathbb{Z}$ is $\mathbb{I}$-local and therefore any identity type in $B\mathbb{Z}$ is $\mathbb{I}$-local. So there is at most one factorisation of any map $\mathbb{I} \to B\mathbb{Z}$ through 1. From $H^1(\mathbb{I}, \mathbb{Z}) = 0$ we get that there merely exists such a factorisation. □

Lemma 6.6.2 Assume X a pointed type such that for all $x : X$ we have $f : \mathbb{I} \to X$ such that $f(0) = *$ and $f(1) = x$. Then X is $\mathbb{I}$-contractible.

Proof For all $x : X$ we get a map:

$$f : \mathbb{I} \to X \to L_{\mathbb{I}}(X)$$

such that $f(0) = [*]$ and $f(1) = [x]$. Since $L_{\mathbb{I}}(X)$ is $\mathbb{I}$-local this means that:

$$\prod_{x:X} [*] = [x]$$

We conclude:

$$\prod_{x:L_{\mathbb{I}}(X)} [*] = x$$

by applying the elimination principle for the modality. □

Corollary 6.6.3 We have that $\mathbb{R}$ and $\mathbb{D}^2 :\equiv \{(x, y) : \mathbb{R} | x^2 + y^2 \le 1\}$ are $\mathbb{I}$-contractible.

Proposition 6.6.4 We have that $L_{\mathbb{I}}(\mathbb{R}/\mathbb{Z}) = B\mathbb{Z}$.

Proof As for any group quotient, the fibers of the map:

$$\mathbb{R} \to \mathbb{R}/\mathbb{Z}$$

are $\mathbb{Z}$-torsor, se we have an induced pullback square:

$$\begin{tikzcd} \mathbb{R} \arrow[r] \arrow[d] & 1 \arrow[d] \\ \mathbb{R}/\mathbb{Z} \arrow[r] & B\mathbb{Z} \end{tikzcd}$$

Now we check that the bottom map is an $\mathbb{I}$-localisation. Since $B\mathbb{Z}$ is $\mathbb{I}$-local by lemma 6.6.1 it is enough to check that its fibers are $\mathbb{I}$-contractible. Since $B\mathbb{Z}$ is connected it is enough to check $\mathbb{R}$ is $\mathbb{I}$-contractible, but this is corollary 6.6.3. □

Remark 6.6.5 By lemma 6.6.1, for any X we have that $H^1(X, \mathbb{Z}) = H^1(L_{\mathbb{I}}(X), \mathbb{Z})$, so that by proposition 6.6.4 we have that $H^1(\mathbb{R}/\mathbb{Z}, \mathbb{Z}) = H^1(B\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$.

We omit the proof that $\mathbb{S}^1 :\equiv \{(x, y) : \mathbb{R} | x^2 + y^2 = 1\}$ is equivalent to $\mathbb{R}/\mathbb{Z}$. The equivalence can be constructed using trigonometric functions, which exists by [BB85][Prop 4.12].

Proposition 6.6.6 The map $\mathbb{S}^1 \to \mathbb{D}^2$ has no retraction.

Proof Otherwise by corollary 6.6.3 and proposition 6.6.4 we would get a retraction of $B\mathbb{Z} \to 1$, so $B\mathbb{Z}$ would be contractible. □

Theorem 6.6.7 (Intermediate value theorem)
For any $f : \mathbb{I} \to \mathbb{I}$ and $y : \mathbb{I}$ such that $f(0) \le y$ and $y \le f(1)$, there exists $x : \mathbb{I}$ such that $f(x) = y$.

Proof By Remark 4.1.2, the proposition $\exists x : \mathbb{I}. f(x) = y$ is closed and therefore $\neg\neg$-stable, so we can proceed with a proof by contradiction. If there is no such $x : \mathbb{I}$, we have $f(x) \neq y$ for all $x : \mathbb{I}$. Then, by Corollary 5.2.10 the following two sets cover $\mathbb{I}$:

$$U_0 :\equiv \{x : \mathbb{I} \mid f(x) < y\} \qquad U_1 :\equiv \{x : \mathbb{I} \mid y < f(x)\}$$

Since U_0 and U_1 are disjoint, we have $\mathbb{I} = U_0 + U_1$ which allows us to define a non-constant function $\mathbb{I} \to 2$, which contradicts Proposition 6.5.3. □

Theorem 6.6.8 (Brouwer's fixed-point theorem)
For all $f : \mathbb{D}^2 \to \mathbb{D}^2$ there exists $x : \mathbb{D}^2$ such that $f(x) = x$.

Proof As above, by Remark 4.1.2, we can proceed with a proof by contradiction, so we assume $f(x) \neq x$ for all $x : \mathbb{D}^2$. For any $x : \mathbb{D}^2$ we set $d_x :\equiv x - f(x)$, so we have that one of the coordinates of d_x is invertible. Let $H_x(t) :\equiv f(x) + t \cdot d_x$ be the line through x and $f(x)$, where "+" and "·" are defined by extending the usual definitions on $\mathbb{I}$. By Theorem 6.6.7 and invertibility of one of the coordinates of d_x, there are intersections of H_x and $\partial\mathbb{D}^2 = \mathbb{S}^1$, both for $t \leq 0$ and for $t > 0$. Since these correspond to distinct solutions of a monic quadratic equation, we know there are exactly two. We denote the intersection for $t > 0$ with $r(x)$, which has the property that it preserves $\mathbb{S}^1$. Then r is a retraction from $\mathbb{D}^2$ onto its boundary $\mathbb{S}^1$, which is a contradiction by Proposition 6.6.6. □

References

[Ásg21] Dagur Ásgeirsson. "The Foundations of Condensed Mathematics". MA thesis. Université de Paris, 2021. URL: https://dagur.sites.ku.dk/condensed-foundations/ (cit. on p. 9).

[BB85] Errett Bishop and Douglas Bridges. *Constructive analysis*. Vol. 279. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1985, pp. xii+477. ISBN: 3-540-15066-8. DOI: 10.1007/978-3-642-61667-9. URL: https://doi.org/10.1007/978-3-642-61667-9 (cit. on pp. 2, 14, 17, 24).

[BC] Reid Barton and Johan Commelin. *lean-ctt-snapshot*. URL: https://github.com/jcommelin/lean-ctt-snapshot (cit. on pp. 2, 9).

[CCH23] Felix Cherubini, Thierry Coquand, and Matthias Hutzler. *A Foundation for Synthetic Algebraic Geometry*. 2023. arXiv: 2307.00073 [math.AG]. URL: https://www.felix-cherubini.de/iag.pdf (cit. on pp. 2, 4).

[CRS21] Thierry Coquand, Fabian Ruch, and Christian Sattler. "Constructive sheaf models of type theory". In: *Math. Struct. Comput. Sci.* 31.9 (2021), pp. 979–1002. DOI: 10.1017/S0960129521000359. URL: https://doi.org/10.1017/S0960129521000359 (cit. on p. 2).

[CS24] Dustin Clausen and Peter Scholze. *Analytic Stacks*. Lecture series. 2023-2024. URL: https://www.youtube.com/playlist?list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO (cit. on pp. 2, 9).

[Die18] Hannes Diener. *Constructive reverse mathematics : Habilitationsschrift*. Fakultät IV - Naturwissenschaftlich-Technische Fakultät, 2018. URL: https://dspace.ub.uni-siegen.de/handle/ubsi/1306 (cit. on pp. 2, 4, 6).

[Dyc76] Roy Dyckhoff. *Categorical methods in dimension theory*. English. Categor. Topol., Proc. Conf. Mannheim 1975, Lect. Notes Math. 540, 220-242 (1976). 1976 (cit. on p. 2).

[Esc04] Martín Escardó. "Synthetic Topology: of Data Types and Classical Spaces". In: *Electronic Notes in Theoretical Computer Science* 87 (2004). Proceedings of the Workshop on Domain Theoretic Methods for Probabilistic Processes, pp. 21–156. ISSN: 1571-0661. DOI: https://doi.org/10.1016/j.entcs.2004.09.017. URL: https://www.sciencedirect.com/science/article/pii/S1571066104051357 (cit. on pp. 2, 6).

[Ish06] Hajime Ishihara. "Reverse Mathematics in Bishop's Constructive Mathematics". In: *Philosophia Scientiae* 6 (Sept. 2006), pp. 43–59. DOI: 10.4000/philosophiascientiae.406 (cit. on pp. 2, 4).

[Leš21] Davorin Lešnik. "Synthetic Topology and Constructive Metric Spaces". PhD thesis. 2021. arXiv: 2104.10399 [math.GN]. URL: https://arxiv.org/abs/2104.10399 (cit. on pp. 2, 6, 9).

[Pro13] The Univalent Foundations Program. *Homotopy type theory: Univalent foundations of mathematics*. https://homotopytypetheory.org/book, 2013 (cit. on pp. 2, 4).

[RSS20] Egbert Rijke, Michael Shulman, and Bas Spitters. "Modalities in homotopy type theory". In: *Logical Methods in Computer Science* Volume 16, Issue 1 (Jan. 2020). DOI: 10.23638/LMCS-16(1:2)2020. URL: https://lmcs.episciences.org/6015 (cit. on p. 24).

[Sch19] Peter Scholze. *Lectures on Condensed Mathematics*. 2019. URL: https://people.mpim-bonn.mpg.de/scholze/Condensed.pdf (cit. on p. 9).

[SDR20] Kristina Sojakova, Floris van Doorn, and Egbert Rijke. "Sequential colimits in homotopy type theory". In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2020)*. ACM, New York, [2020] ©2020, p. 14. ISBN: 978-1-4503-7104-9. DOI: 10.1145/3373718.3394801. URL: https://doi.org/10.1145/3373718.3394801 (cit. on pp. 6, 7, 8).

[Shu18] Michael Shulman. "Brouwer's fixed-point theorem in real-cohesive homotopy type theory". In: *Mathematical Structures in Computer Science* 28.6 (2018), pp. 856–941. DOI: 10.1017/S0960129517000147 (cit. on p. 2).

[Wär24] David Wärn. *On internally projective sheaves of groups*. 2024. arXiv: 2409.12835 [math.CT]. URL: https://arxiv.org/abs/2409.12835 (cit. on p. 2).

[Wei24] Niels van der Weide. *The internal languages of univalent categories*. 2024. arXiv: 2411.06636 [math.CT]. URL: https://arxiv.org/abs/2411.06636 (cit. on p. 2).