Cohomology and Synthetic Stone Duality

May 2, 2025

The following is an incomplete draft on work in progress (so far) by ?
We denote localisation at $\mathbb{I}$ by $\int$.

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1 Acyclicity and Čech cohomology

1.1 First cohomology vanish for Stone spaces

First some result on Čech sequences and cohomology. It works in plain HoTT and is certainly not new.

Lemma 1.1.1 Assume given sets S, S' with a surjection:

$$S' \to S$$

and write $T(x)$ its fiber over $X : S$.

Assume given $A : S \to \mathrm{Ab}$ with:

$$\phi : \prod_{x:S} BA(x)$$

and:

$$\alpha : \prod_{x:S} (\phi(x) = *)^{T(x)}$$

If the beginning of the Čech sequence:

$$\prod_{x:S} A(x)^{T(x)} \to \prod_{x:S} A(x)^{T(x)^2} \to \prod_{x:S} A(x)^{T(x)^3}$$

is exact then $\phi = *$.

Proof See definition 1.3.3 for the definition of the maps in the sequence.

We define:

$$g : \prod_{x:S} T(x)^2 \to A(x)$$

by having $g_x(y,z)$ the the element $\alpha_x(y)^{-1} \cdot \alpha_x(z)$. It is easy to check that it is a cocycle in the given sequence, so that by exactness there is $f : \prod_{x:S} T(x) \to A(x)$ such that for all $x : S$ and $y, z : T(x)$ we have that:

$$g_x(y,z) = f_x(y) \cdot f_x(z)^{-1}$$

Then we define:

$$\alpha' : \prod_{x:S} (\phi(x) = *)^{T(x)}$$

$$\alpha'_x(y) = \alpha_x(y) \cdot f_x(y)$$

so that for all $x : S$ and $y, z : T(x)$ we have that:

$$\alpha'_x(z)^{-1} \cdot \alpha'_x(y) = f_x(z)^{-1} \alpha_x(z)^{-1} \alpha_x(y) \cdot f_x(y) = \mathrm{refl}$$

so that:

$$\alpha'_x(y) = \alpha'_x(z)$$

and this means that α' factors through S, giving a proof of $\phi = *$. ☐

We prove the Čech sequence of a map to a finite type is exact:

Lemma 1.1.2 Assume F finite type with a type X and a surjective map $q : X \to F$. We write $I(x)$ the fibers of q over $x : F$.

Assume given $A : F \to \mathrm{Ab}$, then the Čech sequence:

$$\prod_{x:F} A(x)^{I(x)} \to \prod_{x:F} A(x)^{I(x)^2} \to \prod_{x:F} A(x)^{I(x)^3}$$

is exact.

Proof Since F is finite and exactness is stable under finite products, we can assume $F = 1$, then A is just an abelian group and I an inhabited type, say with $0 : I$. Then we are considering the sequence:

$$A^I \to A^{I^2} \to A^{I^3}$$

with the maps:

$$(a_i)_{i:I} \mapsto (a_i - a_j)_{i,j:I}$$

$$(a_{i,j})_{i,j:I} \mapsto (a_{i,j} + a_{j,k} - a_{i,k})_{i,j,k:I}$$

Assume $(a_{i,j})_{i,j:I}$ such that for all $i, j, k : I$ we have that $a_{i,j} + a_{j,k} = a_{i,k}$.

First we have that $a_{i,i} + a_{i,i} = a_{i,i}$ so that $a_{i,i} = 0$. Then $a_{i,j} + a_{j,i} = a_{i,i}$ gives that $a_{i,j} = -a_{j,i}$. Now we consider $(a_{i,0})_{i:I}$, which is sends to $(a_{i,0} - a_{j,0})_{i,j:I} = (a_{i,j})_{i,j:I}$. So the sequence is indeed exact. ☐

We want to prove that the vanishing of the Čech sequence of Stone spaces with overtly discrete coefficients is stable under sequential limit. It will rely heavily on Scott continuity. We need an auxiliary definition:

Definition 1.1.3 Assume given $S = \lim_k S_k$ sequential limits of Stone and $A : S \to \mathrm{Ab}_{\mathsf{ODisc}}$. Then we define:

$$A_k : S_k \to \mathrm{Ab}_{\mathsf{ODisc}}$$

$$A_k(x) = \prod_{y:S, y|_k = x} A(y)$$

By the dual to Tychonov, we have that $A_k(x)$ is overtly discrete.

Moreover we for all $k : \mathbb{N}$ and $x : S_{k+1}$ we have a map:

$$A_k(x|_k) \to A_{k+1}(x)$$

The two references in the proof are to variants of Scott continuity, in the directed univalence draft.

Lemma 1.1.4 Assume given $S = \lim_k S_k$ and $S' = \lim_k S'_k$ limits of Stone spaces, and a surjective map of towers:

$$\begin{array}{ccccccc}
S'_0 & \longleftarrow & S'_1 & \longleftarrow & S'_2 & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
S_0 & \longleftarrow & S_1 & \longleftarrow & S_2 & \longleftarrow & \cdots
\end{array}$$

We denote the fibers of the induced map $S' \to S$ over $x : S$ by $T(x)$ and the fibers of the map $S'_k \to S_k$ over $x : S_k$ by $T_k(x)$.

Assume given $A : S \to \mathrm{Ab}_{\mathsf{ODisc}}$ such that for all $k : \mathbb{N}$ the Čech sequence:

$$\prod_{x:S_k} A_k(x)^{T_k(x)} \to \prod_{x:S_k} A_k(x)^{T_k(x)^2} \to \prod_{x:S_k} A_k(x)^{T_k(x)^3}$$

is exact. Then the Čech sequence:

$$\prod_{x:S} A(x)^{T(x)} \to \prod_{x:S} A(x)^{T(x)^2} \to \prod_{x:S} A(x)^{T(x)^3}$$

is exact.

Proof We prove that for any $l : \mathbb{N}$ we have that:

$$\mathrm{colim}_k \left(\prod_{x:S_k} A_k(x)^{T_k(x)^l} \right) = \prod_{x:S} A(x)^{T(x)^l}$$

We conclude from this and exactness being stable under sequential colimit. We omit the verification that this is compatible with maps.

We have that:

$$\prod_{x:S_k} A_k(x)^{T_k(x)^l} = \prod_{x:S_k} \prod_{y:S, y|_k = x} A(y)^{T_k(x)^l} = \prod_{y:S} A(y)^{T_k(y|_k)^l}$$

and by **??** we have that:

$$\mathrm{colim}_k \left(\prod_{y:S} A(y)^{T_k(y|_k)^l} \right) = \prod_{y:S} \mathrm{colim}_k \left(A(y)^{T_k(y|_k)^l} \right)$$

but by **??** we have that:

$$\mathrm{colim}_k \left(A(y)^{T_k(y|_k)^l} \right) = A(y)^{\lim_k T_k(y|_k)^l}$$

and we can see by commutation of limits that:

$$\lim_k T_k(y|_k)^l = T(y)^l$$

□

Now we just have to assemble the pieces.

Lemma 1.1.5 Let S be Stone and $A : S \to \mathrm{Ab}_{\mathsf{ODisc}}$. Then:

$$H^1(S, A) = 0$$

Proof Assume given:

$$\phi : \prod_{s:S} BA(x)$$

we have that:

$$\prod_{x:S} \|\phi(x) = *\|$$

so that by local choice there merely exists S' Stone and a surjective map:

$$q : S' \to S$$

with fibers denoted $T(x)$ and:

$$\alpha : \prod_{x:S'} (\phi(x) = *)^{T(x)}$$

In order to conclude by applying lemma 1.1.1, it is enough to prove that the Čech sequence:

$$\prod_{x:S} A(x)^{T(x)} \to \prod_{x:S} A(x)^{T(x)^2} \to \prod_{x:S} A(x)^{T(x)^3}$$

is exact.

By **??** we know that any surjective map between Stone spaces is a sequential limit of surjective maps between finite types. So by applying lemma 1.1.4 we see it is enough to prove the exactness of the exact sequence for maps between finite types, and this is lemma 1.1.2. □

1.2 Stone spaces are acyclic

We extend the previous section to the following: for all S Stone,$A : S \to \mathrm{Ab}_{\mathsf{ODisc}}$ and $n > 0$ we have that:

$$H^n(S, A) = 0$$

We (Thierry & Hugo) follow David's proof in SAG to go from 1 to all n.

Lemma 1.2.1 Assume for all S Stone and $A : \mathrm{Ab}_{\mathsf{ODisc}}$ we have that:

$$H^k(S, A) = 0$$

for all $0 < k < n$. Then:

- For all S Stone and A overtly discrete abelian group, for all $k < n$ the map:

  $$K(A^S, k) \to K(A, k)^S$$

  is an equivalence.
- For all S Stone and A overtly discrete abelian group, the map:

  $$K(A^S, n) \to K(A, n)^S$$

  is an embedding.

Proof We proceed by induction on n. If $n = 0$ this is clear as we always have that $K(A^S, 0) = K(A, 0)^S$.

Assume it holds for n, then:

- We need to prove that the embedding:

  $$K(A^S, n) \to K(A, n)^S$$

  is an equivalence. If $n = 0$ it is immediate and otherwise by $H^n(S, A) = 0$ we know that $K(A, n)^S$ is connected, so the embedding is surjective and therefore an equivalence.
- We need to prove that the map:

  $$K(A^S, n + 1) \to K(A, n + 1)^S$$

  is an embedding, since the source is connected it is enough to prove that:

  $$\Omega K(A^S, n + 1) \to \Omega(K(A, n + 1)^S)$$

  is an equivalence but this is the previous bullet-point. □

Theorem 1.2.2

Let S be Stone and $A : S \to \mathrm{Ab}_{\mathsf{ODisc}}$. Then for all $n > 0$ we have that:

$$H^n(S, A) = 0$$

Proof We proceed by induction on k. For $k = 1$ it is lemma 1.1.5.

Assume it hold for all $0 < k < n$ we want to prove it for n. Assume:

$$\alpha : \prod_{x:S} K(A_x, k+1)$$

By local choice we have a surjective map:

$$f : \sum_{x:S} T_x \to S$$

with T_x Stone such that we merely have:

$$\prod_{x:S} T_x \to (\alpha(x) = *)$$

This means that the image of α under the diagonal map:

$$\prod_{x:S} K(A_x, k+1) \to \prod_{x:S} K(A_x, k+1)^{T_x}$$

merely is zero.

By lemma 1.2.1 it means that the image of α by the map:

$$\prod_{x:S} K(A_x, n) \to \prod_{x:S} K(A_x^{T_x}, n)$$

merely is 0, which means that the map:

$$H^n(x : S, A_x) \to H^n(x : S, A_x^{T_x})$$

sends α to 0.

Then we consider the exact sequence depending on $x : S$:

$$0 \to A_x \to A_x^{T_x} \to L_x \to 0$$

so we have an exact sequence:

$$H^{n-1}(x : S, L_x) \to H^n(x : S, A_x) \to H^n(x : S, A_x^{T_x})$$

where α is send to 0 in $H^n(x : S, A_x^{T_x})$. By induction we have $H^{n-1}(x : S, L_x)$ from which we conclude that α is equal to 0 in $H^n(x : S, A_x)$. □

Corollary 1.2.3 For all S Stone, $A : \mathrm{Ab}_{\mathsf{ODisc}}$ and n we have that the map:

$$K(A^S, n) \to K(A, n)^S$$

is an equivalence.

Proof From lemma 1.2.1 and theorem 1.2.2. □

1.3 Čech cohomology

Definition 1.3.1 A Čech cover for a type X consists of a surjective map:

$$f : S \to X$$

where S is Stone and for all $x : X$ the fiber S_x of f over x is Stone.

Next lemma show that cohomology interact well with Čech cover. It will be used later to prove that cohomology and Čech cohomology agree.

Lemma 1.3.2 Assume given X a type with a Čech cover:

$$f : S \to X$$

as well as $A : X \to \mathrm{Ab}_{\mathsf{ODisc}}$.

For all $n \geq 1$ we have an exact sequence:

$$H^{n-1}(X, x \mapsto A_x^{S_x}) \to H^{n-1}(X, L) \to H^n(X, A) \to 0$$

natural in A, where $L_x = A_x^{S_x}/A_x$.

Proof Using the long exact sequence associated to:

$$0 \to A_x \to A_x^{S_x} \to L_x \to 0$$

by theorem 1.2.2 it is enough to prove that for all n we have:

$$H^n(x : X, A_x^{S_x}) = H^n(x : S, A_x)$$

But by corollary 1.2.3 we have that:

$$\prod_{x:X} K(A_x^{S_x}, n) = \prod_{x:S} K(A_x, n)$$

□

Definition 1.3.3 Assume given a Čech cover

$$f : S \to X$$

and $A : X \to \mathrm{Ab}_{\mathsf{ODisc}}$.

Then we define the Čech complex by:

$$\prod_{x:X} A_x^{S_x} \to \prod_{x:X} A_x^{S_x \times S_x} \to \cdots$$

with the boundary maps defined as expected, that is:

$$\delta(\alpha)(x, u_0, \cdots, u_n) = \sum_{i=0}^{n} (-1)^i \alpha(x, u_0, \cdots, \hat{u_i}, \cdots, u_n)$$

Then the Čech cohomology:

$$\check{H}^k(x : X, A_x)$$

is defined as the k-th homology group of the Čech complex.

Lemma 1.3.4 Assume given a Čech cover:

$$f : S \to X$$

If we are given a short exact sequence of overtly discrete abelian group:

$$0 \to A_x \to B_x \to C_x \to 0$$

depending on $x : X$, there is a long exact sequence of Čech cohomology groups:

$$\check{H}^0(X, A) \to \check{H}^0(X, B) \to \check{H}^0(X, C) \to \check{H}^1(X, A) \to \check{H}^1(X, B) \to \check{H}^1(X, C) \to \cdots$$

Moreover this long exact sequence is natural in the short exact sequence.

Proof We just use the fact that all elements $\sum_{x:X} T_x^{k+1}$ in the Čech complex are Stone spaces, so a short exact sequence of overtly discrete abelian group induces a short exact of Čech complexes by theorem 1.2.2. □

Lemma 1.3.5 Assume given a Čech cover:

$$f : S \to X$$

and $A : X \to \mathrm{Ab}_{\mathsf{ODisc}}$.

For all $n \geq 1$ we have an exact sequence:

$$\check{H}^{n-1}(X, x \mapsto A_x^{S_x}) \to \check{H}^{n-1}(X, L) \to \check{H}^n(X, A) \to 0$$

natural in A where $L_x = A_x^{S_x}/A_x$.

Proof It is enough to prove $\check{H}^n(X, x \mapsto A_x^{S_x}) = 0$ for all $n \geq 1$. Indeed assume given:

$$\alpha : \prod_{x:X} S_x^{n+1} \to S_x \to A_x$$

such that $\delta(\alpha) = 0$, i.e. for all $x : X$ and $u_0, \cdots, u_{n+1}, v : S_x$ we have that:

$$\sum_{i=0}^{n+1} (-1)^i \alpha(x, u_0, \cdots, \hat{u_i}, \cdots, u_{n+1}, v) = 0$$

Then we define:

$$\beta : \prod_{x:X} S_x^n \to S_x \to A_x$$

$$\beta(x, u_0, \cdots, u_{n-1}, v) = (-1)^n \alpha(x, u_0, \cdots, u_{n-1}, v, v)$$

and then:

$$\begin{aligned} \delta(\beta)(x, u_0, \cdots, u_n, v) &= (-1)^n \sum_{i=0}^{n} (-1)^i \alpha(x, u_0, \cdots, \hat{u_i}, \cdots, u_n, v, v) \\ &= \alpha(x, u_0, \cdots, u_n, v) \end{aligned}$$

□

Theorem 1.3.6
Assume given a Čech cover:

$$f : S \to X$$

and $A : X \to \mathrm{Ab}_{\mathsf{ODisc}}$.

Then we have a natural isomorphism:

$$H^n(X, A) = \check{H}^n(X, A)$$

Proof We proceed by induction on n. For $n = 0$ we need to prove that maps:

$$\alpha : \prod_{s:S} A_{f(s)}$$

such that whenever $f(s) = f(t)$ we have that $\alpha(s) = \alpha(t)$ are naturally isomorphic to:

$$\prod_{x:X} A_x$$

This is immediate.

For the inductive step we use lemma 1.3.2 and lemma 1.3.5. Naturality comes from the naturality of the exact sequences. □

1.4 The unit interval is acyclic

Proposition 1.4.1 For all A overtly discrete and all k we have that:

$$H^k(\mathbb{I}, A) = 0$$

Proof TODO □

2 Shape modality

Here we start studying the shape modality $\int$, which is defined as the localisation at $\mathbb{I}$.

2.1 Deloopings of overtly discrete abelian groups are local

Lemma 2.1.1 For all A overtly discrete abelian group and any $n : \mathbb{N}$, we have that:

$$B_n A$$

is $\int$-local.

Proof We proceed by induction on n.

- For $n = 0$ we have $A = A^{\mathbb{I}}$, which holds as maps in $\mathbb{I} \to A$ factor through a finite type and maps in $\mathbb{I} \to 2$ are constant.
- For $n + 1$, by induction hypothesis we know that $B_{n+1}A$ is $\int$-separated, and we merely have a lift by proposition 1.4.1. □

Corollary 2.1.2 Let X be a type and A be an overtly discrete abelian group. Then for all k we have:

$$H^k(X, A) \simeq H^k(\int X, A)$$

2.2 The shape of the circle is the circle

Proposition 2.2.1 We have that:

$$\int(\mathbb{R}/\mathbb{Z}) = B\mathbb{Z}$$

Proof The fibers of the map:

$$\mathbb{R} \to \mathbb{R}/\mathbb{Z}$$

are $\mathbb{Z}$-torsors, as is the case for any group quotient. This means that we have a fiber sequence:

$$\mathbb{R} \to \mathbb{R}/\mathbb{Z} \to B\mathbb{Z}$$

We check that the second map is $\int$-localisation. We have that $B\mathbb{Z}$ is $\int$-local by lemma 2.1.1. Since $B\mathbb{Z}$ is connected we just need to prove that $\mathbb{R}$ is $\int$-contractible to conclude. But $0 : \mathbb{R}$ and for any $x : \mathbb{R}$ there is $f : \mathbb{I} \to \mathbb{R}$ such that $f(0) = 0$ and $f(1) = x$ so we can conclude. □

2.3 Finite homotopical cell complex are local

Lemma 2.3.1 Let X be a finite homotopical cell complex, then for any $x : X$ and any n we have that $\pi_n(X, x)$ is a countably presented abelian group.

Proof TODO maybe find a reference? □

Proposition 2.3.2 Let X be a finite homotopical cell complex, then X is $\int$-local.

Proof We decompose X as its Postnikov tower:

$$\cdots \to \|X\|_{n+1} \to \|X\|_n \to \cdots \to \|X\|_0$$

First we show by induction on n then $\|X\|_n$ is $\int$-local:

- We have that $\|X\|_0$ is a finite set so it is $\int$-local.
- Assuming $\|X\|_n$ is $\int$-local, it is enough to prove that the fibers of the map:

  $$\|X\|_{n+1} \to \|X\|_n$$

  are $\int$-local. But they merely are of the form $B_n\pi_n(X, x)$ for some $x : X$, but $\pi_n(X, x)$ is overtly discrete by lemma 2.3.1 so that $B_n\pi_n(X, x)$ is $\int$-local by lemma 2.1.1.

Therefore the limit of the Postnikov tower is $\int$-local as a limit of $\int$-local group, and we can conclude as a finite homotopical CW complex X is the limit of its Postnikov tower (why TODO, maybe optimistic?).□

Remark 2.3.3 By Anel / Barton "Choice axioms and Postnikov completeness" we know that Postnikov completion and hypercompletion agree in our setting because we have countable choice. Do we have hypercompleteness?

2.4 Cellular cohomology for finite topological cell complex

Definition 2.4.1 An n-dimensional topological cell complex is defined inductively as a type X such that:

- If $n = 0$ then X is a finite type.
- For $n + 1$, we ask that there merely exists X_n an n-dimensional topological cell complex and a pushout square:

$$\begin{tikzcd} X_n \arrow[r] & X \\ \mathbb{S}^n \times \mathrm{Fin}(k) \arrow[u] \arrow[r] & \mathbb{D}^n \times \mathrm{Fin}(k) \arrow[u] \end{tikzcd}$$

A finite topological cell complex is a type that is an n-dimensional topological cell complex for some n.

By contrast we call the usual HoTT cell complexes homotopical.

Lemma 2.4.2 For all n we have that:

$$\int \mathbb{D}^n = 1$$

$$\int \mathbb{S}^n = S^n$$

Proof For any $x : \mathbb{D}^n$ we have a map $f : \mathbb{I} \to \mathbb{D}^n$ such that $f(0) = 0$ and $f(1) = x$ se we have that $\int \mathbb{D}^n = 1$.

For S^n we proceed inductively:

$$\mathbb{S}^{-1} = S^{-1} = 0$$

which is $\int$-local.

Otherwise assume $\int \mathbb{S}^n = S^n$. We have a pushout diagram:

$$\begin{tikzcd} \mathbb{D}^n \arrow[r] & \mathbb{S}^{n+1} \\ \mathbb{S}^n \arrow[u] \arrow[r] & \mathbb{D}^n \arrow[u] \end{tikzcd}$$

which is $\int$-equivalent to the pushout square:

$$\begin{tikzcd} 1 \arrow[r] & S^{n+1} \\ S^n \arrow[u] \arrow[r] & 1 \arrow[u] \end{tikzcd}$$

so that we have:

$$\int \mathbb{S}^{n+1} = \int S^{n+1}$$

but S^{n+1} is $\int$-local by proposition 2.3.2. □

Lemma 2.4.3 Let X be a type such that $\int X$ is a finite homotopical n-dimensional cell complex. Assume given a pushout square:

$$\begin{tikzcd} X \arrow[r] & Y \\ \mathbb{S}^n \times \mathrm{Fin}(k) \arrow[u] \arrow[r] & \mathbb{D}^n \times \mathrm{Fin}(k) \arrow[u] \end{tikzcd}$$

Then we have a pushout square:

$$\begin{tikzcd} \int X \arrow[r] & \int Y \\ S^n \times \mathrm{Fin}(k) \arrow[u] \arrow[r] & \mathrm{Fin}(k) \arrow[u] \end{tikzcd}$$

Proof By general reasoning on modalities and lemma 2.4.2 we have that:

$$\int Y = \int \left(\int X \coprod_{S^n \times \mathrm{Fin}(k)} \mathrm{Fin}(k) \right)$$

but since:

$$\int X \coprod_{S^n \times \mathrm{Fin}(k)} \mathrm{Fin}(k)$$

is a finite homotopical cell complex by hypothesis, is is $\int$-local by proposition 2.3.2 and we can conclude.□

Lemma 2.4.4 Let X be a finite topological cell complex, then $\int X$ is a finite homotopical cell complex. Moreover we can compute a presentation for $\int X$ from a presentation for X simply by localising.

Proof We apply lemma 2.4.3 repeatedly. □

Remark 2.4.5 Given a finite topological cell complex, defining the corresponding finite homotopical cell complex is not obvious, as we need to show the result is independent from the chosen presentation. Using $\int$ allows to bypass this issue.

Corollary 2.4.6 Assume given X a finite topological cell complex and A an overtly discrete abelian group. Then:

$$H^n(X, A)$$

can be computed using the cellular cohomology of the finite homotopical complex $\int X$.

Proof Just recall that by corollary 2.1.2 we have that $H^n(X, A) = H^n(\int X, A)$ and conclude by lemma 2.4.4.□

3 Random Facts

Compilation of various facts on synthetic stone duality which I (Hugo) don't know where to put. I'm not even sure where I should put this section.

3.1 Stone space are sequential limits of finite types

Lemma 3.1.1 Let X be a type. TFAE:

(i) X is Stone.

(ii) X is a sequential limit of finite types.

Proof The key remark is that:

$$\mathrm{Spec}(\mathrm{colim}_k B_k) = \lim_k \mathrm{Spec}(B_k)$$

- (i) implies (ii). We know that a countably presented algebra is a sequential colimit of finitely presented algebras. But spectrum of a f.p. algebra is a finite type and we conclude using the key remark.
- (ii) implies (i). By the key remark and the fact that c.p. algebras are stable by sequential colimits, we conclude that Stone spaces are stable by sequential limit and the fact that finite types are Stone spaces is enough to conclude. □

3.2 Overtly discrete types are sequential colimits of finite types

Lemma 3.2.1 Any open in $\mathbb{N}$ is countable in the sense that it is merely equivalent to a decidable in $\mathbb{N}$.

Proof Assume given a map $U : \mathbb{N} \to \text{Open}$. By countable choice there is a map:

$$\alpha : \mathbb{N} \to \mathbb{N}_\infty$$

such that:

$$\prod_{n:\mathbb{N}} U(n) = (\sum_{k:\mathbb{N}} \alpha(n,k) = 1)$$

Then:

$$\sum_{n:\mathbb{N}} U(n) = \sum_{n,k:\mathbb{N}} \alpha(n,k) = 1$$

which allows us to conclude. □

Lemma 3.2.2 Let X be a type, TFAE:

(i) X is overtly discrete.

(ii) X is a sequential colimit of finite types.

Proof

- (i) implies (ii). Assume X overtly discrete, by using lemma 3.2.1 we know is of the form:

  $$X = (\Sigma_{\mathbb{N}} D)/R$$

  with D decidable and R open. Using choice for $\Sigma_{\mathbb{N}} D$ we get:

  $$\alpha : (\Sigma_{\mathbb{N}} D) \to (\Sigma_{\mathbb{N}} D) \to 2$$

  such that:

  $$R(x,y) = \exists_{k:\mathbb{N}} \alpha(x,y,k) = 1$$

  Then we define:

  $$X_n = (\Sigma_{\text{Fin}(n)} D)/L$$

  $$L(x,y) = \exists_{k:\text{Fin}(n)} \alpha(x,y,k) = 1$$

  We have that X_n is a finite type as it is a decidable quotient of a decidable subset of a finite type. Moreover:

  $$\text{colim}_n X_n = X$$

  as sequential colimit commutes with quotients by equivalence relations.

- (ii) implies (i). Indeed consider a sequential colimit of:

  $$f_k : \text{Fin}(l_k) \to \text{Fin}(l_{k+1})$$

  Then:

  $$\text{colim}_k \text{Fin}(l_k) = \left(\sum_{k:\mathbb{N}} \text{Fin}(l_k) \right) /L$$

  where L is the equivalence relation generated by $(k,x) \sim (k+1, f_k(x))$. But $\sum_{k:\mathbb{N}} \text{Fin}(l_k)$ is a decidable in $\mathbb{N}$ and the equivalence relation generated by a decidable relation on such a type is open. □

3.3 Overtly discrete boolean algebras

Lemma 3.3.1 Let B be a boolean algebra, TFAE:

(i) B is countably presented.

(ii) B is a colimit of finitely presented algebras.

(iii) The underlying type of B is overtly discrete.

Proof We will prove (i) implies (ii) implies (iii) implies (i).

- (i) implies (ii) is known.
- (ii) implies (iii) is an immediate consequence of lemma 3.2.2.
- (iii) implies (i). Assume $B = (\Sigma_{\mathbb{N}} D)/R$ with. D decidable and R open. Then B is the boolean algebra $2[\mathbb{N}]$ generated by $\mathbb{N}$ quotiented by L generated by:
  - For all $x : \mathbb{N}$ such that $\neg D(x)$ we have that $L(x, 0)$.
  - For all $s, t : 2[\mathbb{N}]$ such that $s, t \in 2[\Sigma_{\mathbb{N}} D]$ and $R([s], [t])$ we have that $L(s, t)$.

  This family of relations is indexed by an open in $\mathbb{N}$, and therefore can be indexed by a decidable E in $\mathbb{N}$. It is therefore equivalent to the set of relations indexed by $x : \mathbb{N}$ with the dummy relation $0 = 0$ when $\neg E(x)$. $\square$

Remark 3.3.2 By a similar reasoning, we probably have that abelian groups are overtly discrete if and only is they are countably presented.

Corollary 3.3.3 Given X compact Hausdorff and C_x a c.p. algebra depending on X, we have that:

$$\prod_{x:X} C_x$$

is a c.p. algebra.

Proof By Tychonov, it is overtly discrete. $\square$

3.4 Overt stone spaces

Proposition 3.4.1 Let S be a stone space. TFAE:

(i) S is overt, i.e. for all $U : X \to \text{Open}$ we have that $\exists_{x:S} U(x)$ is an open proposition.

(ii) For all $C : X \to \text{Closed}$ we have that $\forall_{x:S} C(x)$ is closed.

(iii) Equality in 2^S is decidable.

Proof We prove (i) implies (ii) implies (iii) implies (i).

- (i) implies (ii). Given $C : X \to \text{Closed}$ we have that:

  $$\forall_{x:S} C(x) = \neg \exists_{x:S} \neg C(x)$$

  as $C(x)$ is $\neg\neg$-stable, and we can conclude using overtness.
- (ii) implies (iii). We have that (ii) implies equality in 2^S closed, but it is always open so we can conclude.
- (iii) implies (i). Given $U : X \to \text{Open}$ we have that $U = \exists_{n:\mathbb{N}} U_n$ with U_n decidable. We have that:

  $$\neg \exists_{x:S} U_n(x) = \forall_{x:S} \neg U_n(x)$$

  which is decidable by (iii). Therefore by Markov (TODO is this correct Markov?):

  $$\neg(\forall_{n:\mathbb{N}} \neg(\exists_{x:S} U_n(x))) = \exists_{n:\mathbb{N}} \neg\neg(\exists_{x:S} U_n(x)) = \exists_{n:\mathbb{N}} \exists_{x:S} U_n(x) = \exists_{x:S} U(x)$$

  So we have that $\exists_{x:S} U(x)$ is indeed open. $\square$