Notes on cohomology in homotopy type theory

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Introduction

I had help from Urs Schreiber during my phd thesis on understanding the ingredients of the construction of the Mayer-Vietories sequence given below. What is written in section 1 is known in the HoTT-community, with the possible exception of Section 1.4 and Lemma 1.5. In 2013 Michael Shulman wrote a blog post with the ideas this article builds upon. Floris van Doorn actually proved those ideas to work and mechanized them in his phd thesis. They speak of parametrized spectra, which is more general than what we need here.

1 Cohomology of sheaves on types

1.1 HoTT prerequisites

The symbol $\equiv$ is used for definitional equality, the symbol $=$ for equality of objects. A pointed type is a pair $(X, *)$. We suppress the pair when denoting pointed types and just write X and assume that the point is called $*$. We will denote the fiber of map $f : A \to B$ to a pointed type B by

$$f^{-1}(*) :\equiv \sum_{x:A}(f(x) = *)$$

For a pointed type X, we use the abbreviation

$$\Omega X :\equiv (* =_X *)$$

and call ΩX the loop space of X. The type $\|\Omega X\|_0$ is a group. For a map $f : A \to B$ and $x, y : A$, there is an induced map

$$f : (x =_A y) \to (f(x) =_B f(y))$$

A pointed map is a map $f : A \to B$ between pointed types A, B together with an equality $f(*) =_B *$. In this case we have an induced map on loop spaces:

$$\Omega f : \Omega A \to \Omega B$$

We will frequently use Univalent Foundations Program 2013, Lemma 8.4.4, which states:

Proposition 1.1

For any map $f : A \to B$ of pointed types, we have the *fibration sequence*

$$\cdots \to \Omega^2 B \to \Omega f^{-1}(*) \to \Omega A \to \Omega B \to f^{-1}(*) \to A \to B$$

where the map from $\Omega A \to \Omega B$ is $(\Omega f)^{-1}$, i.e. Ωf composed with the map

$$x = y \to y = x.$$

Proposition 1.2

(a) For any group G there is a pointed type BG such that $\Omega BG = G$.

(b) If G is abelian and then for any $n : \mathbb{N}$ there is a pointed type B^nG such that $\Omega^n B^n G = G$. B^nG is also called $K(G,n)$.

(c) The construction extends to maps, i.e. for any group homomorphism $f : G \to H$ there is a pointed map $Bf : BG \to BH$, such that $\Omega Bf = f$.

(d) If $f : A \to B$ is a homomorphism of abelian groups, then $B^n f$ can be constructed.

(i) and (ii) are constructed in Licata and Finster 2014, Section 5. (i) is constructed as a higher inductive type and (iii) can be established by the recursion/induction principles of higher inductive types. (iv) is analogous to (ii).
By Proposition 1.1 we can construct a delooping of the kernel of a group homorphism:

Proposition 1.3
Let $f : G \to H$ be a homomorphism of groups. Then $f^{-1}(*) = \Omega(Bf)^{-1}(*)$.

This is a direct application of Proposition 1.1.
The following will be useful to keep in mind:
Let $f, g : \prod_{x:X} Y_x$ be dependent functions. Univalence implies function extensionality, i.e. the map

$$(f = g) \to \left(\prod_{x:X}(f(x) = g(x))\right)$$

pointwise induced by evaluation is an equivalence. If $(x : X) \mapsto A_x$ is pointwise pointed, $\prod_{x:X} A_x$ is pointed by the map $(x : X) \mapsto *$ and

$$\left(\Omega \prod_{x:X} A_x\right) = \left(\prod_{x:X} \Omega A_x\right)$$

We will also need the following generalization:

Proposition 1.4
Assume for any $x : X$ there is a pullback square

$$\begin{array}{ccc} P_x & \longrightarrow & A_x \\ \downarrow & & \downarrow \\ B_x & \longrightarrow & C_x \end{array}$$

Then

$$\begin{array}{ccc} \prod_{x:X} P_x & \longrightarrow & \prod_{x:X} A_x \\ \downarrow & & \downarrow \\ \prod_{x:X} B_x & \longrightarrow & \prod_{x:X} C_x \end{array}$$

is a pullback square.

1.2 Sheaves

We will use the fact, that sheaves can be represented as maps from their étalé spaces. By univalence, this implies, that a sheaf $\mathcal{F}$ on a type X can be represented as a dependent type

$$(x : X) \mapsto \mathcal{F}_x$$

We will assume throughout that our sheaves are *abelian*, which means that each $\mathcal{F}_x$ is an abelian group. This entails that we can deloop pointwise using Proposition 1.2, i.e. if $(x : X) \mapsto \mathcal{F}_x$ is an abelian sheaf, then for every $n : \mathbb{N}$ there is a sheaf

$$(x : X) \mapsto B^n \mathcal{F}_x$$

For a map $f : Y \to X$ and a sheaf $\mathcal{F}$ on X we have a sheaf

$$f^* \mathcal{F} :\equiv \big((x : Y) \mapsto F_{f(x)}\big)$$

We call a map

$$\varphi \equiv (x : X) \mapsto \varphi_x : \prod_{x:X} (\mathcal{F}_x \to \mathcal{G}_x)$$

of abelian sheaves $\mathcal{F}$ and $\mathcal{G}$ a *homomorphism*, if all φ_x are group homomorphisms.

1.3 Cohomology

This section essentially repeats parts of (van Doorn 2018, Section 5.4) for a special case. In (van Doorn 2018, Section 5) there are results about spectral sequences, which we will not mention here. The cohomology of a sheaf $\mathcal{F}$ on X is defined by

$$H^n(X, \mathcal{F}) :\equiv \left\| \prod_{x:X} B^n \mathcal{F}_x \right\|_0$$

Then $H^n(X, \mathcal{F})$ is an abelian group since

$$H^n(X, \mathcal{F}) = \left\| \prod_{x:X} \Omega^2 B^{n+2} \mathcal{F}_x \right\|_0 = \left\| \Omega^2 \prod_{x:X} B^{n+2} \mathcal{F}_x \right\|_0$$

And we can always view cohomology groups as k-th homotopy groups in the sense of Univalent Foundations Program 2013, Chapter 8:

$$H^n(X, \mathcal{F}) = \left\| \Omega^k \prod_{x:X} B^{n+k} \mathcal{F}_x \right\|_0 = \pi_k(\prod_{x:X} B^{n+k} \mathcal{F}_x)$$

The construction is covariantly functorial in the following sense:
If $\varphi :\equiv (x : X) \mapsto \varphi_x : \prod_{x:X} (\mathcal{F}_x \to \mathcal{G}_x)$ is a homomorphism of sheaves, then there is a homomorphism

$$H^n(X, \varphi) : H^n(X, \mathcal{F}) \to H^n(X, \mathcal{G})$$

The construction is contravariantly functorial in the following sense:
Given $f : Y \to X$, there is a homomorphism:

$$f^* : H^n(X, \mathcal{F}) \to H^n(Y, f^* \mathcal{F})$$

1.4 A long exact sequence

Let us introduce an abbreviation for a sheaf $\mathcal{F} \equiv (x : X) \mapsto \mathcal{F}_x$:

$$\prod \mathcal{F} :\equiv \prod_{x:X} \mathcal{F}_x$$

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of abelian sheaves on a type X. Then there is a sheaf $\mathcal{K}_\varphi$ given as

$$\mathcal{K} :\equiv \mathcal{K}_\varphi :\equiv (x : X) \mapsto \varphi_x^{-1}(*)$$

For any $x : X$ we get a fiber sequence:

$$\cdots \to \Omega^2\mathcal{G}_x \to \Omega\mathcal{K}_x \to \Omega\mathcal{F}_x \to \Omega\mathcal{G}_x \to \mathcal{K}_x \to \mathcal{F}_x \to \mathcal{G}_x$$

By Proposition 1.4 and Proposition 1.3, for any $n : \mathbb{N}$ there is a fiber sequence:

$$\cdots \to \Omega \prod B^n\mathcal{F} \to \Omega \prod B^n\mathcal{G} \to \prod B^n\mathcal{K} \to \prod B^n\mathcal{F} \to \prod B^n\mathcal{G}$$

We can apply Univalent Foundations Program 2013, Theorem 8.4.6 to get an exact sequence (of groups and pointed types)

$$\cdots \to \pi_1\left(\prod B^n\mathcal{F}\right) \to \pi_1\left(\prod B^n\mathcal{G}\right) \to \pi_0\left(\prod B^n\mathcal{K}\right) \to \pi_0\left(\prod B^n\mathcal{F}\right) \to \pi_0\left(\prod B^n\mathcal{G}\right)$$

And therefore an exact sequence of cohomology groups:

$$\cdots \to H^{n-1}(X,\mathcal{F}) \to H^{n-1}(X,\mathcal{G}) \to H^n(X,\mathcal{K}) \to H^n(X,\mathcal{F}) \to H^n(X,\mathcal{G})$$

eventually starting with the proposition $\prod_{x:X} \Omega\mathcal{G}_x$.

1.5 A Mayer-Vietoris lemma

Lemma 1.5
Let $\mathcal{F}$ be an abelian sheaf on X and assume we have a pushout square of spaces

$$\begin{tikzcd} \mathrm{S} \arrow[r, "\varphi_U"] \arrow[d, "\varphi_V"] & U \arrow[d, "\psi_U"] \\ \mathrm{V} \arrow[r, "\psi_V"] & X \end{tikzcd}$$

Then

(a) The square

$$\begin{tikzcd} \prod \varphi_U^* \psi_U^* \mathcal{F} & \prod \psi_U^* \mathcal{F} \arrow[l] \\ \prod \psi_V^* \mathcal{F} \arrow[u] & \prod \mathcal{F} \arrow[l] \arrow[u] \end{tikzcd}$$

is a pullback.

(b) If one of the maps

$$\prod \mathcal{F} \to \prod \psi_U^* \mathcal{F} \text{ and } \prod \mathcal{F} \to \prod \psi_V^* \mathcal{F}$$

is what Myers 2019 calls a $\|\_\|_0$-fibration, then we also have a pullback of cohomology groups:

$$H^n(X, \mathcal{F}) = H^n(U, \psi_U^* \mathcal{F}) \times_{H^n(S, \phi_U^* \psi_U^* \mathcal{F})} H^n(V, \psi_V^* \mathcal{F})$$

(c) We have a Mayer-Vietoris sequence:

$$\to H^{n-1}(S, \varphi_U^* \psi_U^* \mathcal{F}) \to H^n(X, \mathcal{F}) \to H^n(U, \psi_U^* \mathcal{F}) \oplus H^n(V, \psi_V^* \mathcal{F}) \to H^n(S, \varphi_U^* \psi_U^* \mathcal{F})$$

Proof (a) This is (Rijke 2019, Proposition 2.1.6).

(b) By Myers 2019, Theorem 3.5, if we apply a modality with surjective units to a pullback square of which one of the span-maps is a fibration for the modality, then the pullback property is preserved by the modality. $\|\_\|_0$ has surjective units, so the theorem applies.

(c) By Wellen [2017], Lemma 3.3.6, we have a pullback square for each $n : \mathbb{N}$:

$$\begin{tikzcd}
\prod \varphi_U^* \psi_U^* B^n \mathcal{F} \arrow[r] \arrow[d] & 1 \arrow[d] \\
(\prod \varphi_U^* \psi_U^* B^n \mathcal{F}) \times (\prod \varphi_U^* \psi_U^* B^n \mathcal{F}) \arrow[r, "-"] & \prod \varphi_U^* \psi_U^* B^n \mathcal{F}
\end{tikzcd}$$

We rotate and paste a transformed Item (a) from above:

$$\begin{tikzcd}
(\prod \psi_V^* B^n \mathcal{F}) \times (\prod \psi_U^* B^n \mathcal{F}) \arrow[d] & \prod B^n \mathcal{F} \arrow[l] \arrow[d] \\
(\prod \varphi_U^* \psi_U^* B^n \mathcal{F}) \times (\prod \varphi_U^* \psi_U^* B^n \mathcal{F}) \arrow[d, "-"] & \prod \varphi_U^* \psi_U^* B^n \mathcal{F} \arrow[l] \arrow[d] \\
\prod \varphi_U^* \psi_U^* B^n \mathcal{F} & 1 \arrow[l]
\end{tikzcd}$$

Now we take the fiber of the top map:

$$\begin{tikzcd}
1 \arrow[d] & \Omega \prod \varphi_U^* \psi_U^* B^n \mathcal{F} \arrow[l] \arrow[d] \\
(\prod \psi_V^* B^n \mathcal{F}) \times (\prod \psi_U^* B^n \mathcal{F}) \arrow[d] & \prod B^n \mathcal{F} \arrow[l] \arrow[d] \\
(\prod \varphi_U^* \psi_U^* B^n \mathcal{F}) \times (\prod \varphi_U^* \psi_U^* B^n \mathcal{F}) \arrow[d, "-"] & \prod \varphi_U^* \psi_U^* B^n \mathcal{F} \arrow[l] \arrow[d] \\
\prod \varphi_U^* \psi_U^* B^n \mathcal{F} & 1 \arrow[l]
\end{tikzcd}$$

So we get the desired fiber long exact sequence again by taking the long exact sequence of homotopy groups.

References

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