Synthetic $\mathbb{A}^1$-Homotopy Theory

???

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Abstract

The following is a collection of results on $\mathbb{A}^1$ homotopy theory in synthetic algebraic geometry ([CCH23]). Authors so far: Peter Arndt, Felix Cherubini, Hugo Moeneclaey, David Wärn.

Contents

1 $\mathbb{A}^1$-modal Types

Definition 1.0.1 A type X is called $\mathbb{A}^1$-*modal*, if all maps $\gamma : \mathbb{A}^1 \to X$ factor uniquely over 1:

$$\begin{array}{ccc} \mathbb{A}^1 & \xrightarrow{\gamma} & X \\ \downarrow & \overset{\exists!}{\nearrow} & \\ 1 & & \end{array}$$

Definition 1.0.2 Let $\int_{\mathbb{A}^1}$ be the nullification modality at $\mathbb{A}^1$ and $\sigma_X : X \to \int_{\mathbb{A}^1} X$ its unit at a type X.

As a consequence, X is $\mathbb{A}^1$-modal, if and only if, $\int_{\mathbb{A}^1} X = X$.

The following was observed by David Jaz Myers in 2018 for affine schemes of the form $\mathrm{Spec}(R[X]/P)$ for some special polynomials P. We rediscovered this for a similar class of schemes by using surprising results on étale schemes.

Proposition 1.0.3 Let X be a type with decidable equality, then X is $\mathbb{A}^1$-modal. In particular, every separated étale scheme is $\mathbb{A}^1$-modal.

Proof Let $\gamma : \mathbb{A}^1 \to X$. Then $\gamma(0) : X$, so we get $\tilde{\gamma}$ with:

$$\begin{array}{ccccc} \mathbb{A}^1 & \xrightarrow{\quad\gamma\quad} & X & & \\ \| & & \| & & \\ \mathbb{A}^1 & \xrightarrow{\tilde{\gamma}} & 1 + (\prod_{x:X} x \neq \gamma(0)) & \longrightarrow & 2 \\ \downarrow & \nearrow & & & \\ 1 & & & & \end{array}$$

We get the factorization by connectedness of $\mathbb{A}^1$. By [Che+23][Proposition 4.2.10] any separated étale scheme has decidable equality. □

Lemma 1.0.4 Let X be a type such that

$$\prod_{x,y:X} \sum_{\gamma:\mathbb{A}^1 \to X} (\gamma(0) = x) \times (\gamma(1) = y)$$

then $\int X$ is a proposition.

Proof First, using the inverse of the map

$$(\int X) \cong \int X^{\mathbb{A}^1}$$

to construct $(x, y : X) \to \sigma_X(x) = \sigma_X(y)$. By direct applications of the dependent universal property of a uniquely eliminating modality we conclude $(x, y : \int X) \to x = y$. □

Example 1.0.5

(a) Let $(X, *)$ be a pointed type with a multiplicative left action of R, such that for all $x : X$, we have $0 \cdot x = *$ and $1 \cdot x = x$. Then $\int X$ is a proposition by Lemma 1.0.4 and therefore contractible, using the maps

$$\gamma_x :\equiv (r : \mathbb{A}^1) \mapsto r \cdot x : \mathbb{A}^1 \to X.$$

This entails that $\mathbb{D}(n)$, $\mathbb{D}$ and all types with an R-module structure are $\mathbb{A}^1$-connected.

(b) For any pair of types $A, B : \mathcal{U}$, we have the maps

$$f_{A,B} :\equiv (x : \mathbb{A}^1) \mapsto A^{x=0} \times B^{x=1}$$
$$g_{A,B} :\equiv (x : \mathbb{A}^1) \mapsto A \times (x = 0) + B \times (x = 1)$$

both constructions imply with Lemma 1.0.4 that $\int \mathcal{U}$ is a proposition and therefore contractible.

(c) For $A, B : R$-Mod we can use the maps $f_{A,B}$ from above to construct a path where all values carry an R-module structure. Therefore $\int R$-Mod is constractbile as well. The same argument applies to all types of structured types closed under product and exponentiation with propositional affine schemes, e.g. R-Mod_{wqc}.

Definition 1.0.6 Let $\mathbb{A}^\times :\equiv \mathbb{A}^1 \setminus \{0\}$.

To describe $\int_{\mathbb{A}^1} \mathbb{A}^\times$, we will need a construction which is called coreduction, deRham stack, infinitesimal shape or crystalline modality. We will use yet another name:

Definition 1.0.7 For any type X, let $\widetilde{X}$ denote the formally étale [1] replacement of X.

Proposition 1.0.8 $\int_{\mathbb{A}^1} \mathbb{A}^\times = \widetilde{\mathbb{A}^\times}$.

Proof By **??**, the fibers of $\mathbb{A}^\times \to \widetilde{\mathbb{A}^\times}$ are $\int_{\mathbb{A}^1}$-connected, so it is enough to show that $\widetilde{\mathbb{A}^\times}$ is $\int_{\mathbb{A}^1}$-modal. Use Zariski choice in the situation

$$\begin{array}{ccc} & & \mathbb{A}^\times \\ & & \big\downarrow \\ \mathbb{A}^1 & \xrightarrow{\gamma} & \widetilde{\mathbb{A}^\times} \end{array}$$

to get $s_i : D(f_i) \to \mathbb{A}^\times$ with $\neg\neg(s_i = s_j)$ on intersections. Fix $x, y : \mathbb{A}^1$. We will show that $\neg\neg(\gamma(x) = \gamma(y))$. We can assume $s_i = s_j$, so the s_i glue to a map $\mathbb{A}^1 \to \mathbb{A}^\times$, which is a lift of γ. This lift is merely of the form $x \mapsto a_0 + \sum_{i=1}^{n} a_i x^i$ with $a_0 \neq 0$ and nilpotent a_i for $i > 0$, which means $\neg\neg(\gamma(x) = a_0 = \gamma(y))$.

So the original map $\mathbb{A}^1 \to \mathbb{A}^\times$ is weakly constant and therefore constant. □

[1] See [Che+23][Section 5.1]

2 Generalities on the shape modality

2.1 About torsors and modalities

We assume given a modality $\bigcirc$ such that for any X the localisation:

$$X \to \bigcirc X$$

is surjective (e.g. $\mathbb{A}^1$-localisation).

Proposition 2.1.1 Let G be a group. The following are equivalent:

(i) The type BG is $\bigcirc$-modal.

(ii) For all X, any G-torsor over X is $\bigcirc$-étale.

Proof (i) implies (ii). Let $f : X \to Y$ be G-torsor, and $i : A \to B$ an $\bigcirc$-equivalence. Assume given a square:

$$\begin{array}{ccc} A & \xrightarrow{s} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

We want to prove there is a unique dotted lifting.

First we prove that there exists such a lifting. We get a G-torsor over B by pulling back the one over Y. By the commutation of the diagram we know that the torsor is trivial when restricted to A, and since BG is $\bigcirc$-modal and i is an $\bigcirc$-equivalence there is a unique dotted lift in:

$$\begin{array}{cc} A & \longrightarrow BG \\ {\scriptstyle i}\downarrow & \nearrow \\ B & \end{array}$$

So that the torsor is trivial on B as well. From this we get h making the triangle commute:

$$\begin{array}{ccc} & & X \\ & \overset{h}{\nearrow} & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

By the definition of torsors, we know that there is $g : A \to G$ such that for all $x : A$ we have:

$$g(x) \cdot h(i(x)) = s(x)$$

But since G is $\bigcirc$-modal (as BG is) and i is an $\bigcirc$-equivalence, we can lift this map to:

$$\begin{array}{ccc} A & \xrightarrow{g} & G \\ {\scriptstyle i}\downarrow & \underset{g'}{\nearrow} & \\ B & & \end{array}$$

And then $y \mapsto g'(y) \cdot h(y)$ makes the square commutes.

Now assume given h_1, h_2 two such liftings, we consider $g : B \to G$ such that for all $y : B$ we have $g(y) \cdot h_1(y) = h_2(y)$. We know that g has constant value 1 on A, but there is a unique dotted lift in:

$$\begin{array}{ccc} A & \xrightarrow{1} & G \\ \downarrow & \nearrow & \\ B & & \end{array}$$

so g has constant value 1 on all of B and we have $h_1 = h_2$.

(ii) implies (i). Saying that BG is $\bigcirc$-modal means there is a for any $\bigcirc$-connected type X there is a unique dotted lift in any:

$$\begin{array}{ccc} X & \longrightarrow & BG \\ \big\downarrow & \nearrow & \\ 1 & & \end{array}$$

Since any G-torsor is assumed $\bigcirc$-étale, by considering the trivial torsor $G \to 1$ and the fact that $\bigcirc$-étale maps are modal we conclude that G is $\bigcirc$-modal. Then identity types in BG are $\bigcirc$-modal and there is at most one dotted lift.

To show there merely exists a dotted lift, we need to show that any G-torsor $P : X \to BG$ merely is trivial. We know that X is merely inhabited as the localisation:

$$X \to \bigcirc X = 1$$

is assumed to be surjective. Since we want to prove a proposition and we know that X is merely inhabited, and we can assume $x : X$. Since torsors are merely inhabited we can assume $t : P(x)$. Consider:

$$\begin{array}{ccc} 1 & \xrightarrow{(x,t)} & \sum_{x:X} P(x) \\ {\scriptstyle 0}\big\downarrow & \nearrow & \big\downarrow \\ X & \xrightarrow[id]{} & X \end{array}$$

□

The right map is a G-torsor so it is assumed $\bigcirc$-étale, and then we merely have a dotted lift because the left map is an $\bigcirc$-equivalence, and this means that the torsor merely is trivial.

2.2 About $\mathbb{A}^1$-localisation

Lemma 2.2.1 Any $\int_{\mathbb{A}^1}$-connected map is surjective. In particular, for any type X the map:

$$\eta_X : X \to \int_{\mathbb{A}^1} X$$

is surjective.

Proof This holds because any $\int_{\mathbb{A}^1}$-connected type is merely inhabited, as any proposition is $\mathbb{A}^1$-local so that for any X we have a map:

$$\int_{\mathbb{A}^1} X \to \|X\|$$

□

Lemma 2.2.2 If X is path-connected then so is $\int_{\mathbb{A}^1} X$.

Proof Immediate from lemma 2.2.1. □

Lemma 2.2.3 If colimits indexed by I exists in HoTT (e.g. pushouts, sequential colimits, quotients of group actions), and we have a map of I-indexed diagrams:

$$f_i : X_i \to Y_i$$

such that for all $i : I$ the map f_i is a $\int_{\mathbb{A}^1}$-equivalence, then the induced map:

$$\text{colim}_{i:I} X_i \to \text{colim}_{i:I} Y_i$$

is a $\int_{\mathbb{A}^1}$-equivalence.

Proof For any $\mathbb{A}^1$-local type Z, we have that:

$$(\text{colim}_{i:I} Y_i) \to Z \simeq \lim_{i:I}(Y_i \to Z)$$

$$\simeq \lim_{i:I}(X_i \to Z) \simeq (\text{colim}_{i:I} X_i) \to Z$$

which implies what we want. □

We often use this lemma with the $\int_{\mathbb{A}^1}$-equivalences:

$$\eta_X : X \to \int_{\mathbb{A}^1} X$$

Next lemma says that $\mathbb{A}^1$-pullbacks can be computed as plain pullbacks for $\mathbb{A}^1$-étale maps.

Lemma 2.2.4 Assume a pulllback square:

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

where the right map is $\mathbb{A}^1$-étale, then it is an $\mathbb{A}^1$-pullback, meaning the square:

$$\begin{array}{ccc} \int_{\mathbb{A}^1} A & \longrightarrow & \int_{\mathbb{A}^1} X \\ \downarrow & & \downarrow \\ \int_{\mathbb{A}^1} B & \longrightarrow & \int_{\mathbb{A}^1} Y \end{array}$$

is a pullback square.

Proof This is [CR21][Corollary 5.2]. We give an alternative proof. Any such pullback square is of the form:

$$\begin{array}{ccc} \sum_{b:B} P(\int_{\mathbb{A}^1} g(\eta_A(b))) & \longrightarrow & \sum_{y:Y} P(\eta_Y(y)) \\ \downarrow & & \downarrow \\ B & \xrightarrow[g]{} & Y \end{array}$$

for some $P : \int_{\mathbb{A}^1} Y \to \mathcal{U}_{\mathbb{A}^1}$, by [CR21][Corollary 5.5]. Applying $\mathbb{A}^1$-localisation to this square gives:

$$\begin{array}{ccc} \sum_{b:\int_{\mathbb{A}^1} B} P(\int_{\mathbb{A}^1} g(b))) & \longrightarrow & \sum_{y:\int_{\mathbb{A}^1} Y} P(y) \\ \downarrow & & \downarrow \\ \int_{\mathbb{A}^1} B & \xrightarrow[\int_{\mathbb{A}^1} g]{} & \int_{\mathbb{A}^1} Y \end{array}$$

which is a pullback square. □

Corollary 2.2.5 Any fiber sequence:

$$X \to Y \to Z$$

with the second map $\mathbb{A}^1$-étale is an $\mathbb{A}^1$-fiber sequence.

Lemma 2.2.6 A map is $\mathbb{A}^1$-étale if and only if it induces equivalences of $\mathbb{A}^1$-disks.

Proof This is a [CR21][Proposition 3.7], using surjectivity from lemma 2.2.1. □

Lemma 2.2.7 Given a span:

$$A \to B \leftarrow X$$

where A and B are $\mathbb{A}^1$-local, we have an equivalence:

$$\int_{\mathbb{A}^1}(A \times_B X) \simeq A \times_B \int_{\mathbb{A}^1} X$$

Proof Since $A \times_B \int_{\mathbb{A}^1} X$ is $\mathbb{A}^1$-local, as a limit of $\mathbb{A}^1$-local type, it is enough to check that the map:

$$A \times_B X \to A \times_B \int_{\mathbb{A}^1} X$$

has $\mathbb{A}^1$-contractible fibers to conclude. But its fibers are equivalent to fibers of the map:

$$X \to \int_{\mathbb{A}^1} X$$

which are indeed $\mathbb{A}^1$-contractible. □

Remark 2.2.8 This implies that colimits in $\mathbb{A}^1$-local types, which are given by:

$$\mathrm{colim}^{\mathbb{A}^1}_{i:I} X_i = \int_{\mathbb{A}^1} \mathrm{colim}_{i:I} X_i$$

are universal, despite $\mathbb{A}^1$-local types not forming a topos for lack of a universe. This holds for any localisation, in fact it holds for any orthogonal factorisation system where the left class is stable by pullback.

3 $\mathbb{A}^1$-coverings

The aim here is to copy the abstract covering theory in [CR21][Section 8], using the $\mathbb{A}^1$-shape defined further down. We cite results from it without proofs. We import the notion of $\bigcirc$-étale maps for a modality $\bigcirc$ from [CR21].

Definition 3.0.1 Let $\bigcirc$ be a modality (in the sense of the HoTT-Book), then $f : X \to Y$ is $\bigcirc$-*étale*, if the naturality square is a pullback:

$$\begin{array}{ccc} X & \longrightarrow & \bigcirc X \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \longrightarrow & \bigcirc Y \end{array}$$

Moreover, a map $g : X \to Y$ is called $\bigcirc$-*equivalence*, if $\bigcirc g$ is an equivalence. The ($\bigcirc$-equivalences, $\bigcirc$-étale maps) is an orthogonal factorization system.

Definition 3.0.2 Let $\int_n :\equiv \int_{\mathbb{A}^1,n}$ be the nullification at $\mathbb{A}^1$ and S^{n+1}.

Definition 3.0.3 An $\mathbb{A}^1$-covering of X is a $\int_1$-étale maps to X which fibers are sets.

Lemma 3.0.4 The type of $\mathbb{A}^1$-covering of X is equivalent to families of $\mathbb{A}^1$-local sets over $\int_{\mathbb{A}^1}$.

Remark 3.0.5 For any $\mathbb{A}^1$-covering $Y \to X$ we have the following lifting property:

$$\begin{array}{ccc} 1 & \longrightarrow & Y \\ \downarrow & \overset{\exists!}{\nearrow} & \downarrow \\ \mathbb{A}^1 & \longrightarrow & X \end{array}$$

which gives perhaps clearer geometric intuitions.

Definition 3.0.6 The *universal cover* of a pointed type X is the pointed map $\hat{X}$ obtained by pullback:

$$\begin{array}{ccc} \hat{X} & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & \int_1 X \end{array}$$

Proposition 3.0.7 Let X be a type, then $\hat{X} \to X$ is indeed an $\mathbb{A}^1$-covering. It is initial among pointed $\mathbb{A}^1$-covering.

Proposition 3.0.8 A pointed $\mathbb{A}^1$-covering $Y \to X$ is universal if and only if $\int_1 Y = 1$.

Proof Since an $\mathbb{A}^1$-covering is $\int_1$-étale we have a pullback square:

$$\begin{array}{ccc} Y & \longrightarrow & \int_1 Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & \int_1 X \end{array}$$

so if $\int_1 Y = 1$ it is indeed the universal covering.

Conversely the pullback square:

$$\begin{array}{ccc} \hat{X} & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & \int_1 X \end{array}$$

as a $\int_1$-étale right map, so its $\int_1$-localisation is still a pullback and we can conclude that $\int_1 \hat{X} = 1$. $\square$

Proposition 3.0.9 Assume $G \to Y \to X$ is a fiber sequence with the right map being a universal cover. Then we have:

$$\pi_1^{\mathbb{A}^1}(X) = G$$

where $\pi_1^{\mathbb{A}^1}(X)$ is defined as $\Omega(\int_1 X)$.

Proof Compose the pullback squares:

$$\begin{array}{ccccc} G & \longrightarrow & Y & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & X & \longrightarrow & \int_1 X \end{array}$$

□

Last result can be generalised to non-universal $\mathbb{A}^1$-coverings.

Corollary 3.0.10 Let:

$$G \to X \to Y$$

be a pointed $\mathbb{A}^1$-covering with X being such that $\int_1 X$ is 0-connected. Then we have an exact sequence of groups:

$$0 \to \pi_1^{\mathbb{A}^1}(X) \to \pi_1^{\mathbb{A}^1}(Y) \to G \to 0$$

Proof We have a fiber sequence:

$$G \to \int_1 X \to \int_1 Y$$

where G is a set, so we have a long fiber sequence:

$$0 \to \Omega \int_1 X \to \Omega \int_1 Y \to G \to \int_1 X$$

We can conclude by set-truncating. □

4 Shape of projective spaces

Proposition 4.0.1 We have that:

$$\int_{\mathbb{A}^1} \mathbb{P}^1 \simeq \int_{\mathbb{A}^1} \mathrm{Susp}(\mathbb{A}^\times)$$

Proof We apply the lemma 2.2.3 to the pushout diagram:

$$\begin{array}{ccc} \mathbb{A}^\times & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

□

Lemma 4.0.2 For any natural number n and any $V : B\mathbb{A}^\times$ we have a $\int_{\mathbb{A}^1}$-equivalence:

$$V^n \setminus \{0\} \to (V^\times)^{*n}$$

where $(V^\times)^{*n}$ is the n-th iterated join of $V^\times$. Moreover these maps are natural in n.

Proof It is clear for $n = 0$ or 1. Inductively we can apply lemma 2.2.3 to the $\int_{\mathbb{A}^1}$-equivalences from the pushout diagram:

$$\begin{array}{ccc} V^\times \times V^n \setminus \{0\} & \longrightarrow & V \times V^n \setminus \{0\} \\ \downarrow & & \downarrow \\ V^\times \times V^n & \longrightarrow & V^{n+1} \setminus \{0\} \end{array}$$

to the pushout diagram:

$$\begin{array}{ccc} V^\times \times (V^\times)^{*n} & \longrightarrow & 1 \times (V^\times)^{*n} \\ \downarrow & & \downarrow \\ V^\times \times 1 & \longrightarrow & (V^\times)^{*n+1} \end{array}$$

Naturality is a straightforward induction. □

Proposition 4.0.3 For any n we have that:

$$\int_{\mathbb{A}^1} \mathbb{P}^n \simeq \int_{\mathbb{A}^1} \sum_{V:B\mathbb{A}^\times} (V^\times)^{*n+1}$$

Proof Using the fact that:

$$\mathbb{P}^n = \sum_{V:B\mathbb{A}^\times} V^{n+1} \setminus \{0\}$$

with lemma 2.2.3 and lemma 4.0.2. □

One might try to compute the previous type more explicitely using the following result:

Lemma 4.0.4 For any n we have a pushout diagram:

$$\begin{array}{ccc} (\mathbb{A}^\times)^{*n} & \longrightarrow & \sum_{V:B\mathbb{A}^\times} (V^\times)^{*n} \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \sum_{V:B\mathbb{A}^\times} (V^\times)^{*n+1} \end{array}$$

where the top map is the natural inclusion.

Proof For any $V : B\mathbb{A}^\times$, we have a pushout diagram:

$$\begin{array}{ccc} V^\times \times (V^\times)^{*n} & \longrightarrow & 1 \times (V^\times)^{*n} \\ \downarrow & & \downarrow \\ V^\times \times 1 & \longrightarrow & (V^\times)^{*n+1} \end{array}$$

So that we have a pushout diagram:

$$\begin{array}{ccc} \sum_{V:B\mathbb{A}^\times} V^\times \times (V^\times)^{*n} & \longrightarrow & \sum_{V:B\mathbb{A}^\times} (V^\times)^{*n} \\ \downarrow & & \downarrow \\ \sum_{V:B\mathbb{A}^\times} V^\times & \longrightarrow & \sum_{V:B\mathbb{A}^\times} (V^\times)^{*n+1} \end{array}$$

But we have that:

$$V^\times \simeq (V =_{R\text{-Mod}} R)$$

so that:

$$(\sum_{V:B\mathbb{A}^\times} V^\times) \simeq 1$$

$$(\sum_{V:B\mathbb{A}^\times} V^\times \times (V^\times)^{*n}) \simeq (\mathbb{A}^\times)^{*n}$$

and we can conclude. □

Lemma 4.0.5 For any $V : B\mathbb{A}^\times$, we have that:

$$\mathrm{colim}_{n:\mathbb{N}} (V^\times)^{*n}$$

is contractible.

Proof Since we want to prove a property, we may assume $V = R$. Since $R^\times$ is inhabited by 1, for all n the map:

$$(V^\times)^{*n} \to (V^\times)^{*(n+1)}$$

is constant and their sequential colimit is contractible. □

Proposition 4.0.6 We have that:

$$\int_{\mathbb{A}^1} \mathbb{P}^\infty \simeq \int_{\mathbb{A}^1} B\mathbb{A}^\times$$

Proof We consider:

$$\mathbb{P}^\infty = \mathrm{colim}_{n:\mathbb{N}} \sum_{V:B\mathbb{A}^\times} V^{n+1} \setminus \{0\}$$

but this is equivalent to:

$$\sum_{V:B\mathbb{A}^\times} \mathrm{colim}_{n:\mathbb{N}}(V^{n+1} \setminus \{0\})$$

which, by lemma 4.0.2 and lemma 2.2.3 is $\int_{\mathbb{A}^1}$-equivalent to:

$$\sum_{V:B\mathbb{A}^\times} \mathrm{colim}_{n:\mathbb{N}}(V^\times)^{*(n+1)}$$

but by lemma 4.0.5 this is equivalent to $B\mathbb{A}^\times$. □

Remark 4.0.7 This is actually part of a larger story, where starting from a higher group G with a delooping BG, we construct for any $n : \mathbb{N}$ a fiber sequence:

$$G \to S^n_G \to P^n_G$$

where:

$$S^n_G = G^{*n}$$

$$P^n_G = \sum_{V:BG} (V =_{BG} *)^{*n}$$

Then we have that:

$$S^\infty_G = 1$$

$$P^\infty_G = BG$$

and that P^{n+1}_G is the cofiber of the map:

$$S^n_G \to P^n_G$$

Starting from the group S^0 this gives the real projective spaces, from S^1 this gives the complex projective spaces, and from $\mathbb{A}^\times$ this gives something $\int_{\mathbb{A}^1}$-equivalent to the fiber sequences:

$$\mathbb{A}^\times \to \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$$

5 Shape of grassmanians

Definition 5.0.1 For any R-module V, we define $F_n(V)$ as the type of surjective maps in:

$$\mathrm{Hom}_{R\text{-Mod}}(R^n, V)$$

Remark 5.0.2 We identify BGL_k with the type:

$$\sum_{V:R\text{-Mod}} \|V = R^k\|$$

Definition 5.0.3 We define the grassmanians:

$$Gr_{n,k} = \sum_{V:BGL_k} F_n(V)$$

A link to another part where the grassmanians are already defined should probably be added...

Definition 5.0.4 We define:

$$F_\infty(V) = \mathrm{colim}_{n:\mathbb{N}} F_n(V)$$

where we use the maps:

$$F_n(V) \to F_{n+1}(V)$$

given by precomposing with the projection forgetting the last scalar:

$$R^{n+1} \to R^n$$

Definition 5.0.5 We define:

$$Gr_{\infty,k} = \text{colim}_{n:\mathbb{N}} Gr_{n,k}$$

Lemma 5.0.6 The maps:

$$F_n(R^k) \to F_{n+k}(R^k)$$

given by precomposing with the projection:

$$p : R^{n+k} \to R^n$$

is $\mathbb{A}^1$-homotopic to a constant map.

Proof For any $t : \mathbb{A}^1$ and $f : \text{Hom}_{R\text{-Mod}}(R^n, R^k)$, we define:

$$g : \text{Hom}_{R\text{-Mod}}(R^n \oplus R^k, R^k)$$

$$g(x, y) = tf(x) + (1-t)y$$

We check that if f is surjective then so is g. But since we want to prove a property we can assume that t is invertible or $1-t$ is invertible, and it is easy to conclude in both cases.

So this indeed gives a map:

$$H : \mathbb{A}^1 \times F_n(R^k) \to F_{n+k}(R^k)$$

Checking that $H(0, f) = f \circ p$ and that $H(1, f)$ does not depend on f is straightforward. □

Lemma 5.0.7 For any V in BGL_k, we have that:

$$\int_{\mathbb{A}^1} F_\infty(V) = 1$$

Proof Since we want to prove a proposition we can assume that $V = R^k$. Then we have that:

$$F_\infty(R^k) = \text{colim}_{n:\mathbb{N}} F_{nk}(R^k)$$

but by lemma 5.0.6 the $\mathbb{A}^1$-localisation of the connecting maps:

$$F_{nk}(R^k) \to F_{nk+k}(R^k)$$

are constant, so that by lemma 2.2.3 we have that $F_\infty(R^k)$ is $\mathbb{A}^1$-equivalent to a sequential colimit of constant maps, i.e a contractible type. □

Proposition 5.0.8 We have that:

$$\int_{\mathbb{A}^1} Gr_{\infty,k} = \int_{\mathbb{A}^1} BGL_k$$

Proof As sequential colimits commute with dependent sums, we have that:

$$Gr_{\infty,k} \simeq \sum_{V:BGL_k} F_\infty(V)$$

but by lemma 5.0.7 and lemma 2.2.3, we know that this is $\mathbb{A}^1$-equivalent to BGL_k. □

6 Jouanolou's trick

The goal here is to prove that any quasi-projective scheme is $\mathbb{A}^1$-equivalent to an affine scheme. We roughly follow [Homotopy Algebraic K-theory by Weibel].

Definition 6.0.1 An affine $\mathbb{A}^1$-replacement for a scheme X consists of an affine scheme W with an $\mathbb{A}^1$-connected map:

$$W \to X$$

Typical affine $\mathbb{A}^1$-replacement are vector bundles or torsors over a vector bundle.

Lemma 6.0.2 There exists an affine $\mathbb{A}^1$-replacement for $\mathbb{P}^n$.

Proof Consider W the type of projection of R^{n+1} of rank 1. There is a map:

$$p : W \to \mathbb{P}^n$$

sending a projection to its image.

The type W is an affine scheme because it is equivalent to the type of square $n+1$ matrices M such that $M^2 = M$ and M characteristic polynomial is $X^n(X-1)$.

Now we need to check that the fibers of p are $\mathbb{A}^1$-connected. Since giving a projection is equivalent to giving its image and its kernel, any fiber of p is merely equivalent to the type of complements for a line in R^{n+1}. So all fibers are merely equivalent and we can just check that the fiber over $[1 : 0 : \cdots : 0]$ is $\mathbb{A}^1$-connected. This fiber is the type of matrices where the first line is of the form $(1, a_1, \cdots, a_n)$ and the rest is 0. This is equivalent to $\mathbb{A}^n$ which is indeed $\mathbb{A}^1$-connected. □

Lemma 6.0.3 Let $p : X \to Y$ be an affine map between schemes. Then the pullback of an affine $\mathbb{A}^1$-replacement for Y along p is an affine $\mathbb{A}^1$-replacement for X.

Proof Immediate, as affine schemes are closed under dependent product. □

Lemma 6.0.4 Let $U \subset \mathrm{Spec}(A)$ be an open subscheme of an affine scheme. Then there exists an affine $\mathbb{A}^1$-replacement for U.

Proof Assume U is of the form $D(f_1) \cup \cdots \cup D(f_n)$. Then we consider affine scheme:

$$M = \{x : \mathrm{Spec}(A), y_1, \cdots, y_n : R \mid f_1(x)y_1 + \cdots + f_n(x)y_n = 1\}$$

There is a canonical projection map from M to U, using the fact that if $f_1(x)y_1 + \cdots + f_n(x)y_n = 1$ then one of the $f_i(x)$ is non-zero.

We just need to prove that the fibers of this map are $\mathbb{A}^1$-connected. But assume $x : \mathrm{Spec}(A)$ such that say $f_j(x) \neq 0$, we see that the fiber over x is equivalent to $\mathbb{A}^{n-1}$, as the equation:

$$f_1(x)y_1 + \cdots + f_n(x)y_n = 1$$

defines y_j as a function of the other y_ks. □

Proposition 6.0.5 Let X be a scheme with an $\mathbb{A}^1$-affine replacement. Then any open or closed subscheme of X has an $\mathbb{A}^1$-replacement.

Proof Given a closed subscheme of X we just apply lemma 6.0.3 and the fact that closed propositions are affine.

Given an open subscheme $U \subset X$, we consider $\mathrm{Spec}(A) \to X$ an affine $\mathbb{A}^1$-replacement and $U' \subset \mathrm{Spec}(A)$ the pullback of U. The map $U' \to U$ is $\mathbb{A}^1$-connected as it is a pullback of the $\mathbb{A}^1$-connected map $\mathrm{Spec}(A) \to X$. By lemma 6.0.4 we have an affine $\mathbb{A}^1$-replacement for U', and we can conclude by using the fact that the composition of $\mathbb{A}^1$-connected maps is $\mathbb{A}^1$-connected. □

Next proposition could be called Jouanolou's trick.

Proposition 6.0.6 Any quasi-projective scheme (defined as closed in open in projective space) merely has an affine $\mathbb{A}^1$-replacement.

Proof We just apply lemma 6.0.2 and proposition 6.0.5. □

Remark 6.0.7 The map $W \to X$ given in the last proposition is surjective and smooth (as it is a composition of maps with fibers merely equivalent to affine spaces), therefore W is smooth if and only if X is smooth. So a smooth quasi-projective scheme has a smooth affine $\mathbb{A}^1$-replacement.

Remark 6.0.8 This result can be extended to any scheme with an ample family of line bundles (known as Jouanolou-Thomason Theorem).

7 What definition to use for motivic homotopy groups?

This section consists of informal notes by Hugo. It is mostly based on skimming through $\mathbb{A}^1$-Algebraic Topology over a Field by Morel.

There are two possible definitions for motivic homotopy groups:

- The naive one:
$$\pi_n^{\mathbb{A}^1}(X) = \pi_n(\textstyle\int_{\mathbb{A}^1} X)$$

- The refined one:
$$\pi_n^{\mathbb{A}^1}(X) = \Omega^n(\textstyle\int_{\mathbb{A}^1, S^{n+1}} X)$$

Traditionally (e.g. In $\mathbb{A}^1$-Algebraic Topology over a Field by Morel), the naive one is used, and a some work is spent proving that it agrees with the refined one in favorable cases, allowing computations. The proof relies crucially on the fact that we have a base field.

This is formulated using three key definitions:

- A set X is $\mathbb{A}^1$-invariant when it is $\mathbb{A}^1$-local.
- A group G is strongly $\mathbb{A}^1$-invariant when BG (and therefore G) is $\mathbb{A}^1$-local.
- An abelian group A is strictly $\mathbb{A}^1$-invariant when $B^n A$ is $\mathbb{A}^1$-local for any $n : \mathbb{N}$.

Then Theorem 1.9 states that for any X we have that:

- The naive $\pi_1^{\mathbb{A}^1}(X)$ is strongly $\mathbb{A}^1$-invariant. I think this implies that if X is $\mathbb{A}^1$-connected then $\|\int_{\mathbb{A}^1} X\|_1$ is $\mathbb{A}^1$-local.
- For any $n > 1$ the naive $\pi_n^{\mathbb{A}^1}(X)$ is strictly $\mathbb{A}^1$-invariant. I think this together with the previous point implies that if X is $\mathbb{A}^1$-connected then $\|\int_{\mathbb{A}^1} X\|_n$ is $\mathbb{A}^1$-local for any $n > 1$, using a Postnikov tower.
- It is conjectured that $\pi_0^{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-invariant. This and the previous two points should imply that the naive and refined definition of motivic homotopy groups agree, again using a Postnikov tower, and probably Whitehead.

The proof relies crucially on the base being a field, and we do not expect both definitions to agree in general.

Remark 7.0.1 Theorem 1.18 states that if X is n-connected, then $\int_{\mathbb{A}^1} X$ is n-connected. It also states that this fails when the base is not a field. When using the refined definition this is seems provable using reasoning on modalities. This seems to contradict both definitions agreeing when the base is not a field.

I believe it would be more fruitful to use the refined definition (as my naming of the definitions subtly suggests...), but an unpleasant consequence of it is that we cannot reuse results from HoTT (long exact sequences, universal covers, ...) directly, they have to be proven again. Maybe this work can be done using any localisation instead of $\mathbb{A}^1$-localisation, making it somewhat reusable?

8 Computing $\pi_1^{\mathbb{A}^1}(\mathbb{P}^n)$ (assuming some hypothesis)

In this section we use $\widetilde{X}$ to denote the formally étale replacement of X. We assume $B\widetilde{\mathbb{A}}^\times$ is $\mathbb{A}^1$-local. We don't know if this holds over any base in the model, but we think this holds over a base field.

Remark 8.0.1 It is enough to assume $H^1(\mathbb{A}^1, \widetilde{\mathbb{A}}^\times) = 0$ or equivalently $H^1(\widetilde{\mathbb{A}}^1, \widetilde{\mathbb{A}}^\times) = 0$. Indeed we $\widetilde{\mathbb{A}}^\times$ is $\mathbb{A}^1$-local so $B\widetilde{\mathbb{A}}^\times$ is $\mathbb{A}^1$-separated, and the cohomology condition means it is $\mathbb{A}^1$-smooth.

Moreover we use the so-called refined definition of motivic homotopy groups, meaning:

$$\pi_n^{\mathbb{A}^1}(X) = \Omega^n(\textstyle\int_n X)$$

Lemma 8.0.2 If X is a smooth scheme then the map:

$$X \to \widetilde{X}$$

is $\mathbb{A}^1$-connected.

Proof Since X is smooth the map:

$$X \to \widetilde{X}$$

is surjective by **??**. So any of its fiber is merely equivalent to $N_\infty(x)$ for some $x : X$. By **??** any such $N_\infty(x)$ is merely equivalent to $N_\infty(0)$ in some $\mathbb{A}^n$, but these are $\mathbb{A}^1$-connected by example 1.0.5. □

Lemma 8.0.3 Assuming that $B\widetilde{\mathbb{A}}^\times$ is $\mathbb{A}^1$-local, we have that for all n the map:

$$\widetilde{\mathbb{A}}^{n+1}/\{0\} \to \widetilde{\mathbb{P}}^n$$

is an $\mathbb{A}^1$-cover with fiber $\widetilde{\mathbb{A}}^\times$.

Proof The considered map is an $\widetilde{\mathbb{A}}^\times$-torsor so we can conclude using proposition 2.1.1. □

Lemma 8.0.4 Assume $n \geq 2$, then we have that $\widetilde{\mathbb{A}}^{n+1}/\{0\}$ is $\mathbb{A}^1$-(n-1)-connected, meaning that:

$$\int_{n-1}(\widetilde{\mathbb{A}}^{n+1}/\{0\}) = 1$$

Proof By lemma 8.0.2 we have that $\widetilde{\mathbb{A}}^{n+1}/\{0\}$ is $\mathbb{A}^1$-equivalent to $\mathbb{A}^{n+1}/\{0\}$, but by lemma 4.0.2 this is $\mathbb{A}^1$-equivalent to $(\mathbb{A}^\times)^{*(n+1)}$. But this type is $(n-1)$-connected as for any pointed type X the join X^{*n} is $n-2$-connected. Therefore it is $\mathbb{A}^1$-$(n-1)$-connected. □

Proposition 8.0.5 Assuming that $B\widetilde{\mathbb{A}}^\times$ is $\mathbb{A}^1$-local, we have that:

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^n) = \widetilde{\mathbb{A}}^\times$$

for all $n \geq 2$ and that:

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^\infty) = \widetilde{\mathbb{A}}^\times$$

Proof First we treat the case of $\mathbb{P}^\infty$. By proposition 4.0.6 we have that $\mathbb{P}^\infty$ is $\mathbb{A}^1$-equivalent to $B\mathbb{A}^\times$, which is itself $\mathbb{A}^1$-equivalent to $B\widetilde{\mathbb{A}}^\times$. So we have that:

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^\infty) = \pi_1^{\mathbb{A}^1}(B\widetilde{\mathbb{A}}^\times)$$

but since $B\widetilde{\mathbb{A}}^\times$ is simply connected and assumed $\mathbb{A}^1$-local, we have that:

$$\pi_1^{\mathbb{A}^1}(B\widetilde{\mathbb{A}}^\times) = \Omega B\widetilde{\mathbb{A}}^\times = \widetilde{\mathbb{A}}^\times$$

Now assume $n \geq 2$. By lemma 8.0.2 we know that:

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^n) = \pi_1^{\mathbb{A}^1}(\widetilde{\mathbb{P}}^n)$$

Then by lemma 8.0.3 we have an $\mathbb{A}^1$-covering:

$$\widetilde{\mathbb{A}}^\times \to \widetilde{\mathbb{A}}^{n+1}/\{0\} \to \widetilde{\mathbb{P}}^n$$

But by lemma 8.0.4 we have that $\widetilde{\mathbb{A}}^{n+1}/\{0\}$ is $\mathbb{A}^1$-1-connected, so by proposition 3.0.8 the cover is universal and by proposition 3.0.9 we can conclude that:

$$\pi_1^{\mathbb{A}^1}(\widetilde{\mathbb{P}}^n) = \widetilde{\mathbb{A}}^\times$$

□

What about $\mathbb{P}^1$?

Proposition 8.0.6 Assuming that $B\widetilde{\mathbb{A}}^\times$ is $\mathbb{A}^1$-local, we have an exact sequence:

$$0 \to \pi_1^{\mathbb{A}^1}(\mathbb{A}^2/\{0\}) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \widetilde{\mathbb{A}}^\times \to 0$$

Proof Very similar to proposition 8.0.5 except we use corollary 3.0.10 instead of proposition 3.0.9, and we need to prove that $\int_1 \mathbb{A}^2/\{0\}$ is 0-connected. But it is $\mathbb{A}^1$-equivalent to $\mathbb{A}^\times * \mathbb{A}^\times$ which is 0-connected, and we know that the map $X \to \int_1 X$ is surjective so $\int_1 \mathbb{A}^2/\{0\}$ is 0-connected as well. □

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